# Alternative Connectives for Classical Propositional Logic

Hans-Dieter Hiep

June 29, 2016

# Syntax: Natural Deduction

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Syntactic derivability: $\Gamma \vdash \psi$
  - a proof tree with conclusion $\psi$ and open assumptions $\Gamma$

**Example: a set of rules (Minimal Logic + Double Negation)**

$$\frac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \to \psi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \phi\end{matrix}}{\psi} \to_e \qquad \frac{}{\neg\neg\phi \to \phi}\ \mathrm{DN}$$

Where $\neg\phi := \phi \to \bot$.

# Syntax: Natural Deduction

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Syntactic derivability: $\Gamma \vdash \psi$
  - a proof tree with conclusion $\psi$ and open assumptions $\Gamma$

**Example: a set of rules (Minimal Logic + Double Negation)**

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \to \psi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \phi\end{matrix}}{\psi} \to_e \qquad \dfrac{}{\neg\neg\phi \to \phi}\ \mathrm{DN}$$

Where $\neg\phi := \phi \to \bot$.

# Syntax: Natural Deduction

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Syntactic derivability: $\Gamma \vdash \psi$
  - a proof tree with conclusion $\psi$ and open assumptions $\Gamma$

**Example: a set of rules (Minimal Logic + Double Negation)**

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \to \psi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \phi\end{matrix}}{\psi} \to_e \qquad \dfrac{}{\neg\neg\phi \to \phi} \text{DN}$$

Where $\neg\phi := \phi \to \bot$.

# Syntax: Natural Deduction

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Syntactic derivability: $\Gamma \vdash \psi$
  - a proof tree with conclusion $\psi$ and open assumptions $\Gamma$

**Example: a set of rules (Minimal Logic + Double Negation)**

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \to \psi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \phi\end{matrix}}{\psi} \to_e \qquad \dfrac{}{\neg\neg\phi \to \phi} \mathrm{DN}$$

Where $\neg\phi := \phi \to \bot$.

# Syntax: Natural Deduction

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Syntactic derivability: $\Gamma \vdash \psi$
  - a proof tree with conclusion $\psi$ and open assumptions $\Gamma$

**Example: a set of rules (Minimal Logic + Double Negation)**

$$\dfrac{\begin{array}{c}[\phi]^u \\ \mathcal{D} \\ \psi\end{array}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}' \\ \phi \to \psi & \phi\end{array}}{\psi} \to_e \qquad \dfrac{}{\neg\neg\phi \to \phi} \text{DN}$$

Where $\neg\phi := \phi \to \bot$.

## Example: Peirce Law

$$\neg\neg p \to p \quad \text{DN}$$

$$\neg\neg p \quad \to_i, v$$

$$p \quad \to_e$$

$$((p \to q) \to p) \to p \quad \to_i, u$$

$$\{\neg p, p\} \vdash q$$
$$\{\neg p\} \vdash p \to q$$
$$\{(p \to q) \to p\} \vdash \neg\neg p$$
$$\emptyset \vdash ((p \to q) \to p) \to p$$

## Example: Peirce Law

$$\dfrac{\neg p \qquad p}{\bot}\ \to_e$$

$$\dfrac{\bot}{\neg\neg p}\ \to_i, v$$

$$\dfrac{\dfrac{}{\neg\neg p \to p}\ \text{DN} \qquad \neg\neg p}{p}\ \to_e$$

$$\dfrac{p}{((p \to q) \to p) \to p}\ \to_i, u$$

$$\{\neg p, p\} \vdash q$$
$$\{\neg p\} \vdash p \to q$$
$$\{(p \to q) \to p\} \vdash \neg\neg p$$
$$\emptyset \vdash ((p \to q) \to p) \to p$$

# Syntax: Natural Deduction

## Example: Peirce Law

$$\dfrac{\dfrac{}{\neg\neg p \to p}\ \mathrm{DN} \qquad \dfrac{\dfrac{\neg p \qquad p}{\bot}\ \to_e}{\neg\neg p}\ \to_i, v}{\dfrac{p}{((p \to q) \to p) \to p}\ \to_i, u}\ \to_e$$

$$\{\neg p, p\} \vdash q$$
$$\{\neg p\} \vdash p \to q$$
$$\{(p \to q) \to p\} \vdash \neg\neg p$$
$$\emptyset \vdash ((p \to q) \to p) \to p$$

## Example: Peirce Law

$$\dfrac{\dfrac{}{\neg\neg p \to p}\ \mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{v} \qquad \dfrac{(p \to q) \to p \qquad p \to q}{p}\ {\to_e}}{\bot}\ {\to_e}}{\neg\neg p}\ {\to_i},\ v}{\dfrac{p}{((p \to q) \to p) \to p}\ {\to_i},\ u}\ {\to_e}$$

$$\{\neg p, p\} \vdash q$$
$$\{\neg p\} \vdash p \to q$$
$$\{(p \to q) \to p\} \vdash \neg\neg p$$
$$\emptyset \vdash ((p \to q) \to p) \to p$$

# Syntax: Natural Deduction

## Example: Peirce Law

$$
\dfrac{
  \dfrac{}{\neg\neg p \to p}\ \mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{
      \neg p^{\,v}
      \qquad
      \dfrac{
        (p \to q) \to p^{\,u}
        \qquad
        \dfrac{
          \dfrac{}{q}
        }{p \to q}\ \to_i, w
      }{p}\ \to_e
    }{\bot}\ \to_e
  }{\neg\neg p}\ \to_i, v
}{
  \dfrac{p}{((p \to q) \to p) \to p}\ \to_i, u
}\ \to_e
$$

$$\{\neg p, p\} \vdash q$$

$$\{\neg p\} \vdash p \to q$$

$$\{(p \to q) \to p\} \vdash \neg\neg p$$

$$\emptyset \vdash ((p \to q) \to p) \to p$$

### Example: Peirce Law

$$\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{q}{p \to q}\,\to_i, w}{p}\,\to_e}{\bot}\,\to_e}{\neg\neg p}\,\to_i, v}{\dfrac{p}{((p \to q) \to p) \to p}\,\to_i, u}\,\to_e$$

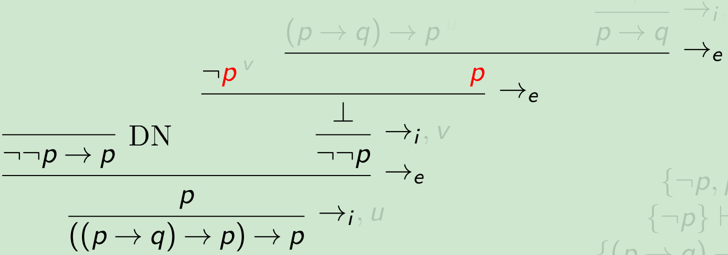
$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law




$$\{\neg p, p\} \vdash q$$
$$\{\neg p\} \vdash p \rightarrow q$$
$$\{(p \rightarrow q) \rightarrow p\} \vdash \neg\neg p$$
$$\emptyset \vdash ((p \rightarrow q) \rightarrow p) \rightarrow p$$

## Example: Peirce Law

$$\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad p \to q}{p}\to_e}{\bot}\to_e}{\neg\neg p}\to_i,\, v}{p}\to_e}{((p \to q) \to p) \to p}\to_i,\, u$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

# Syntax: Natural Deduction

## Example: Peirce Law

$$
\dfrac{
  \dfrac{}{\neg\neg p \to p}\,\mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{
      \neg p^{\,v}
      \qquad
      \dfrac{
        (\color{red}{p \to q}) \to p^{\,u}
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{}{\neg\neg q \to q}\,\mathrm{DN}
            \qquad
            \dfrac{
              \dfrac{\neg p^{\,w} \qquad p^{\,w}}{\bot}\,\to_e
            }{\neg\neg q}\,\to_i
          }{q}\,\to_e
        }{\color{red}{p \to q}}\,\to_i,\, w
      }{p}\,\to_e
    }{\bot}\,\to_e
  }{\neg\neg p}\,\to_i,\, v
}{
  \dfrac{p}{((p \to q) \to p) \to p}\,\to_i,\, u
}\,\to_e
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\to_e}{\neg\neg q}\to_i}{q}\to_e}{p \to q}\to_i, w}{p}\to_e}{\bot}\to_e}{\neg\neg p}\to_i, v}{p}\to_e}{((p \to q) \to p) \to p}\to_i, u$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\to_e}{\neg\neg q}\to_i}{q}\to_e}{p \to q}\to_i, w}{p}\to_e}{\bot}\to_e}{\neg\neg p}\to_i, v}{p}\to_e}{((p \to q) \to p) \to p}\to_i, u$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\,\to_e}{\neg\neg q}\,\to_i}{q}\,\to_e}{p \to q}\,\to_i, w}{p}\,\to_e}{\bot}\,\to_e}{\neg\neg p}\,\to_i, v}{p}\,\to_e}{((p \to q) \to p) \to p}\,\to_i, u$$

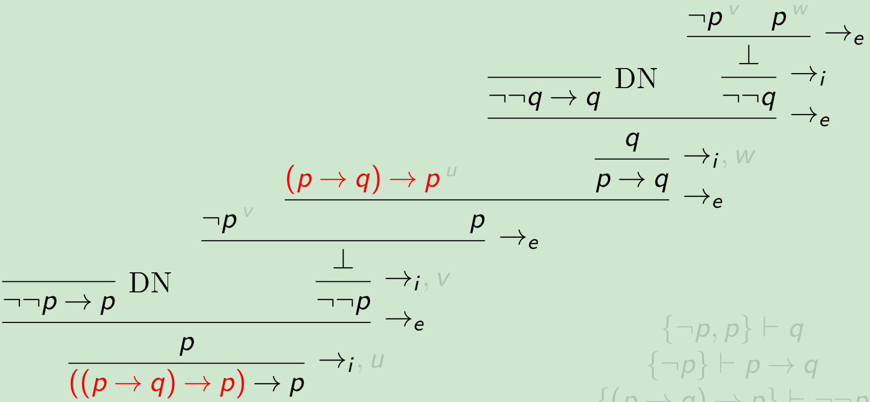
$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law




$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$
\dfrac{
  \dfrac{}{\neg\neg p \to p}\,\mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{
      \neg p^{\,v}
      \qquad
      \dfrac{
        (p \to q) \to p^{\,u}
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{}{\neg\neg q \to q}\,\mathrm{DN}
            \qquad
            \dfrac{
              \dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\,\to_e
            }{\neg\neg q}\,\to_i
          }{q}\,\to_e
        }{p \to q}\,\to_i, w
      }{p}\,\to_e
    }{\bot}\,\to_e
  }{\neg\neg p}\,\to_i, v
}{
  \dfrac{p}{((p \to q) \to p) \to p}\,\to_i, u
}\,\to_e
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

# Syntax: Natural Deduction

## Example: Peirce Law

$$
\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p\to q)\to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q\to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \quad p^{\,w}}{\bot}\,{\to_e}}{\neg\neg q}\,{\to_i}}{q}\,{\to_e}}{p\to q}\,{\to_i,\ w}}{p}\,{\to_e}}{\bot}\,{\to_e}}{\neg\neg p}\,{\to_i,\ v}}{p}\,{\to_e}}{((p\to q)\to p)\to p}\,{\to_i,\ u}
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$
\dfrac{
  \dfrac{}{\neg\neg p \to p}\,\mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{
      \neg p^{\,v}
      \qquad
      \dfrac{
        (p \to q) \to p^{\,u}
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{}{\neg\neg q \to q}\,\mathrm{DN}
            \qquad
            \dfrac{
              \dfrac{\neg p^{\,v} \quad p^{\,w}}{\bot}\,{\to_e}
            }{\neg\neg q}\,{\to_i}
          }{q}\,{\to_e}
        }{p \to q}\,{\to_i, w}
      }{p}\,{\to_e}
    }{\bot}\,{\to_e}
  }{\neg\neg p}\,{\to_i, v}
}{
  \dfrac{p}{((p \to q) \to p) \to p}\,{\to_i, u}
}\,{\to_e}
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$
\dfrac{
  \dfrac{\;}{\neg\neg p \to p}\,\mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{
      \neg p^{\,v}
      \qquad
      \dfrac{
        (p \to q) \to p^{\,u}
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{\;}{\neg\neg q \to q}\,\mathrm{DN}
            \qquad
            \dfrac{
              \dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\to_e
            }{\neg\neg q}\to_i
          }{q}\to_e
        }{p \to q}\to_{i},\, w
      }{p}\to_e
    }{\bot}\to_e
  }{\neg\neg p}\to_{i},\, v
}{
  \dfrac{p}{((p \to q) \to p) \to p}\to_{i},\, u
}\to_e
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

# Syntax: Natural Deduction

## Example: Peirce Law

$$
\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\to_e}{\neg\neg q}\to_i}{q}\to_e}{p \to q}\to_{i}, w}{p}\to_e}{\bot}\to_e}{\neg\neg p}\to_{i}, v}{p}\to_e}{((p \to q) \to p) \to p}\to_{i}, u
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$
\dfrac{
  \dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN}
  \qquad
  \dfrac{
    \dfrac{\neg p^{\,v} \qquad
      \dfrac{(p \to q) \to p^{\,u} \qquad
        \dfrac{
          \dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\,\to_e}{\neg\neg q}\,\to_i}{q}\,\to_e
        }{p \to q}\,\to_i, w
      }{p}\,\to_e
    }{\bot}\,\to_e
  }{\neg\neg p}\,\to_i, v
  }{p}\,\to_e
}{((p \to q) \to p) \to p}\,\to_i, u
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$
\dfrac{\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\,{\to_e}}{\neg\neg q}\,{\to_i}}{q}\,{\to_e}}{p \to q}\,{\to_i, w}}{p}\,{\to_e}}{\bot}\,{\to_e}}{\neg\neg p}\,{\to_i, v}}{p}\,{\to_e}}{((p \to q) \to p) \to p}\,{\to_i, u}
$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$\dfrac{\dfrac{}{\neg\neg p \to p}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad \dfrac{(p \to q) \to p^{\,u} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg q \to q}\,\mathrm{DN} \qquad \dfrac{\dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\to_e}{\neg\neg q}\to_i}{q}\to_e}{p \to q}\to_i, w}{p}\to_e}{\bot}\to_e}{\neg\neg p}\to_i, v}{\dfrac{p}{((p \to q) \to p) \to p}\to_i, u}\to_e$$

$\{\neg p, p\} \vdash q$

$\{\neg p\} \vdash p \to q$

$\{(p \to q) \to p\} \vdash \neg\neg p$

$\emptyset \vdash ((p \to q) \to p) \to p$

## Example: Peirce Law

$$
\dfrac{
  \dfrac{
    \dfrac{}{\neg\neg p \to p}\,\mathrm{DN}
    \qquad
    \dfrac{
      \dfrac{
        \neg p^{\,v}
        \qquad
        \dfrac{
          (p \to q) \to p^{\,u}
          \qquad
          \dfrac{
            \dfrac{
              \dfrac{}{\neg\neg q \to q}\,\mathrm{DN}
              \qquad
              \dfrac{
                \dfrac{\neg p^{\,v} \qquad p^{\,w}}{\bot}\,\to_e
              }{\neg\neg q}\,\to_i
            }{q}\,\to_e
          }{p \to q}\,\to_i, w
        }{p}\,\to_e
      }{\bot}\,\to_e
    }{\neg\neg p}\,\to_i, v
  }{p}\,\to_e
}{((p \to q) \to p) \to p}\,\to_i, u
$$

$$\{\neg p, p\} \vdash q$$
$$\{\neg p\} \vdash p \to q$$
$$\{(p \to q) \to p\} \vdash \neg\neg p$$
$$\emptyset \vdash ((p \to q) \to p) \to p$$

# Semantics: Classical

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Semantic entailment: $\Gamma \models \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if $\Gamma$ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

**Example: Classical Minimal Semantics**

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \to \psi]\!]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and $[\![\bot]\!] := 0$

# Semantics: Classical

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Semantic entailment: $\Gamma \models \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if $\Gamma$ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

## Example: Classical Minimal Semantics

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \to \psi]\!]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and $[\![\bot]\!] := 0$

# Semantics: Classical

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Semantic entailment: $\Gamma \vDash \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if $\Gamma$ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

**Example: Classical Minimal Semantics**

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \to \psi]\!]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and $[\![\bot]\!] := 0$

# Semantics: Classical

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Semantic entailment: $\Gamma \vDash \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if $\Gamma$ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

**Example: Classical Minimal Semantics**

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \to \psi]\!]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and $[\![\bot]\!] := 0$

# Semantics: Classical

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Semantic entailment: $\Gamma \vDash \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if $\Gamma$ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

**Example: Classical Minimal Semantics**

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \to \psi]\!]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and $[\![\bot]\!] := 0$

# Semantics: Classical

- Given some natural deduction system **D**.
- Operates on formulas in some smallest set $\mathcal{F}$.
- Semantic entailment: $\Gamma \models \psi$
  - [for all $\phi \in \Gamma$ where $[\![\phi]\!] = 1$] implies $[\![\psi]\!] = 1$
  - (if $\Gamma$ is empty, $[\![\psi]\!] = 1$ must therefore always hold)

## Example: Classical Minimal Semantics

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \to \psi]\!]$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

and $[\![\bot]\!] := 0$

# Semantics: Classical

## Example: Peirce Law

| $[\![p]\!]$ | $[\![q]\!]$ | $[\![p \to q]\!]$ | $[\![(p \to q) \to p]\!]$ | $[\![((p \to q) \to p) \to p]\!]$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

- $\{q\} \vDash p \to q$,
- $\{p\} \vDash (p \to q) \to p$,
- $\emptyset \vDash ((p \to q) \to p) \to p$.

# Semantics: Classical

## Example: Peirce Law

| $[\![p]\!]$ | $[\![q]\!]$ | $[\![p \to q]\!]$ | $[\![(p \to q) \to p]\!]$ | $[\![((p \to q) \to p) \to p]\!]$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

- $\{q\} \models p \to q$,
- $\{p\} \models (p \to q) \to p$,
- $\emptyset \models ((p \to q) \to p) \to p$.

# Semantics: Classical

## Example: Peirce Law

| $[\![p]\!]$ | $[\![q]\!]$ | $[\![p \to q]\!]$ | $[\![(p \to q) \to p]\!]$ | $[\![((p \to q) \to p) \to p]\!]$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

- $\{q\} \vDash p \to q$,
- $\{p\} \vDash (p \to q) \to p$,
- $\emptyset \vDash ((p \to q) \to p) \to p$.

# Semantics: Classical

## Example: Peirce Law

| $[\![p]\!]$ | $[\![q]\!]$ | $[\![p \to q]\!]$ | $[\![(p \to q) \to p]\!]$ | $[\![((p \to q) \to p) \to p]\!]$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |

- $\{q\} \vDash p \to q$,
- $\{p\} \vDash (p \to q) \to p$,
- $\emptyset \vDash ((p \to q) \to p) \to p$.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \vDash \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vDash \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy,
- Showing completeness is harder.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \models \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \models \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy.
- Showing completeness is harder.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \vDash \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vDash \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy.
- Showing completeness is harder.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \vDash \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vDash \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy.
- Showing completeness is harder.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \models \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \models \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy.
- Showing completeness is harder.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \vDash \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vDash \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy,
- Showing completeness is harder.

# Soundness and Completeness

- Given some subset of formulas, $\Gamma \subseteq \mathcal{F}$
- Soundness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vdash \psi$ implies $\Gamma \vDash \psi$
- Completeness (of a natural deduction system **D** w.r.t. semantic entailment)
  - $\Gamma \vDash \psi$ implies $\Gamma \vdash \psi$

- Showing soundness is (relatively) easy,
- Showing completeness is harder.

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic: $\top, \bot, \neg, \wedge, \vee, \to$, and
    - $\mathcal{F}_M$ for minimal logic: $\bot, \to$
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$.
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

## Example: Minimal Logic

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

## Example: Minimal Logic

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic: $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic: $\bot, \rightarrow$
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

## Example: Minimal Logic

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \to$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \to$.
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

## Example: Minimal Logic

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$.
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$.
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

## Example: Minimal Logic

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$.
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

## Example: Minimal Logic

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$.
  - What does translation mean?

# Outline of Contribution (1)

- Idea: take two natural deduction systems
  - The first, **A**, is known to be sound and complete,
  - Of the second, **B**, it is not known,
  - Show how every proof tree of **B** translates into **A** (soundness),
  - Show how every proof tree of **A** translates into **B** (completeness).

**Example: Minimal Logic**

- It is well-known that minimal logic plus axiom of double negation is sound and complete (cf. Zena Ariola et al.)
- Can we translate standard logic to minimal logic and back?

- Problems:
  - Two natural deduction systems may operate on different formulas,
    - $\mathcal{F}_S$ for standard logic $\top, \bot, \neg, \wedge, \vee, \rightarrow$, and
    - $\mathcal{F}_M$ for minimal logic $\bot, \rightarrow$.
  - What does translation mean?

# Preserving Translations

Let **A** and **B** be two natural deduction systems.

**Definition**

A function $t$ is a *truth-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

**Definition**

A function $t$ is a *provability-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and for all proof trees $\mathcal{D}$ in **A**, there exists a proof tree $\mathcal{D}'$ in **B**, s.t.:

$$\begin{array}{c} [\phi_0, \ldots, \phi_n] \\ \mathcal{D} \\ \psi \end{array} \quad \Rightarrow \quad \begin{array}{c} [t(\phi_0), \ldots, t(\phi_n)] \\ \mathcal{D}' \\ t(\psi) \end{array}$$

# Preserving Translations

Let **A** and **B** be two natural deduction systems.

## Definition

A function $t$ is a *truth-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

## Definition

A function $t$ is a *provability-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and for all proof trees $\mathcal{D}$ in **A**, there exists a proof tree $\mathcal{D}'$ in **B**, s.t.:

$$\begin{array}{c} [\phi_0, \ldots, \phi_n] \\ \mathcal{D} \\ \psi \end{array} \implies \begin{array}{c} [t(\phi_0), \ldots, t(\phi_n)] \\ \mathcal{D}' \\ t(\psi) \end{array}$$

# Preserving Translations

Let **A** and **B** be two natural deduction systems.

## Definition

A function $t$ is a *truth-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

## Definition

A function $t$ is a *provability-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and for all proof trees $\mathcal{D}$ in **A**, there exists a proof tree $\mathcal{D}'$ in **B**, s.t.:

$$\begin{array}{c} [\phi_0, \ldots, \phi_n] \\ \mathcal{D} \\ \psi \end{array} \quad \Rightarrow \quad \begin{array}{c} [t(\phi_0), \ldots, t(\phi_n)] \\ \mathcal{D}' \\ t(\psi) \end{array}$$

# Preserving Translations

Let **A** and **B** be two natural deduction systems.

### Definition

A function $t$ is a *truth-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

### Definition

A function $t$ is a *provability-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and for all proof trees $\mathcal{D}$ in **A**, there exists a proof tree $\mathcal{D}'$ in **B**, s.t.:

$$\begin{array}{c} [\phi_0, \ldots, \phi_n] \\ \mathcal{D} \\ \psi \end{array} \quad \Rightarrow \quad \begin{array}{c} [t(\phi_0), \ldots, t(\phi_n)] \\ \mathcal{D}' \\ t(\psi) \end{array}$$

# Preserving Translations

Let **A** and **B** be two natural deduction systems.

**Definition**

A function $t$ is a *truth-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

**Definition**

A function $t$ is a *provability-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and for all proof trees $\mathcal{D}$ in **A**, there exists a proof tree $\mathcal{D}'$ in **B**, s.t.:

$$\begin{array}{c} [\phi_0, \ldots, \phi_n] \\ \mathcal{D} \\ \psi \end{array} \quad \Rightarrow \quad \begin{array}{c} [t(\phi_0), \ldots, t(\phi_n)] \\ \mathcal{D}' \\ t(\psi) \end{array}$$

# Preserving Translations

Let **A** and **B** be two natural deduction systems.

## Definition

A function $t$ is a *truth-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and their valuations are always equal, i.e. $[\![\phi]\!] = [\![t(\phi)]\!]$.

## Definition

A function $t$ is a *provability-preserving* translation function if and only if:

- given some formula $\phi$ of **A**, the image $t(\phi)$ is a formula of **B**,
- and for all proof trees $\mathcal{D}$ in **A**, there exists a proof tree $\mathcal{D}'$ in **B**, s.t.:

$$\begin{matrix}[\phi_0, \ldots, \phi_n] \\ \mathcal{D} \\ \psi\end{matrix} \quad \Rightarrow \quad \begin{matrix}[t(\phi_0), \ldots, t(\phi_n)] \\ \mathcal{D}' \\ t(\psi)\end{matrix}$$

# Preserving Translations

**Example: Minimal Formulas**

Let $\mathcal{F}_M$ be the smallest set s.t.:

$$a \in \mathcal{F}_M,$$
$$\bot \in \mathcal{F}_M,$$
$$\phi \to \psi \in \mathcal{F}_M,$$

where $\phi, \psi \in \mathcal{F}_M$.

N.b. minimal formulas directly map to classical formulas.

**Example: Standard Formulas**

Let $\mathcal{F}_S$ be the smallest set s.t.:

$$a \in \mathcal{F}_S,$$
$$\bot \in \mathcal{F}_S,$$
$$\top \in \mathcal{F}_S,$$
$$\neg\phi \in \mathcal{F}_S,$$
$$\phi \wedge \psi \in \mathcal{F}_S,$$
$$\phi \vee \psi \in \mathcal{F}_S,$$
$$\phi \to \psi \in \mathcal{F}_S,$$

where $\phi, \psi \in \mathcal{F}_S$.

# Preserving Translations

**Example: Minimal Formulas**

Let $\mathcal{F}_M$ be the smallest set s.t.:

$$a \in \mathcal{F}_M,$$
$$\bot \in \mathcal{F}_M,$$
$$\phi \to \psi \in \mathcal{F}_M,$$

where $\phi, \psi \in \mathcal{F}_M$.

N.b. minimal formulas directly map to classical formulas.

**Example: Standard Formulas**

Let $\mathcal{F}_S$ be the smallest set s.t.:

$$a \in \mathcal{F}_S,$$
$$\bot \in \mathcal{F}_S,$$
$$\top \in \mathcal{F}_S,$$
$$\neg\phi \in \mathcal{F}_S,$$
$$\phi \wedge \psi \in \mathcal{F}_S,$$
$$\phi \vee \psi \in \mathcal{F}_S,$$
$$\phi \to \psi \in \mathcal{F}_S,$$

where $\phi, \psi \in \mathcal{F}_S$.

# Preserving Translations

**Example: Minimal Formulas**

Let $\mathcal{F}_M$ be the smallest set s.t.:

$$a \in \mathcal{F}_M,$$
$$\bot \in \mathcal{F}_M,$$
$$\phi \to \psi \in \mathcal{F}_M,$$

where $\phi, \psi \in \mathcal{F}_M$.

N.b. minimal formulas directly map to classical formulas.

**Example: Standard Formulas**

Let $\mathcal{F}_S$ be the smallest set s.t.:

$$a \in \mathcal{F}_S,$$
$$\bot \in \mathcal{F}_S,$$
$$\top \in \mathcal{F}_S,$$
$$\neg\phi \in \mathcal{F}_S,$$
$$\phi \wedge \psi \in \mathcal{F}_S,$$
$$\phi \vee \psi \in \mathcal{F}_S,$$
$$\phi \to \psi \in \mathcal{F}_S,$$

where $\phi, \psi \in \mathcal{F}_S$.

# Preserving Translations

**Example: Minimal Formulas**

Let $\mathcal{F}_M$ be the smallest set s.t.:

$$a \in \mathcal{F}_M,$$
$$\bot \in \mathcal{F}_M,$$
$$\phi \to \psi \in \mathcal{F}_M,$$

where $\phi, \psi \in \mathcal{F}_M$.

N.b. minimal formulas directly map to classical formulas.

**Example: Standard Formulas**

Let $\mathcal{F}_S$ be the smallest set s.t.:

$$a \in \mathcal{F}_S,$$
$$\bot \in \mathcal{F}_S,$$
$$\top \in \mathcal{F}_S,$$
$$\neg\phi \in \mathcal{F}_S,$$
$$\phi \wedge \psi \in \mathcal{F}_S,$$
$$\phi \vee \psi \in \mathcal{F}_S,$$
$$\phi \to \psi \in \mathcal{F}_S,$$

where $\phi, \psi \in \mathcal{F}_S$.

## Preserving Translations

### Example: Standard to Minimal Translation

Let $t : \mathcal{F}_S \to \mathcal{F}_M$ be:

$$
\begin{aligned}
t(\top) &:= \bot \to \bot, \\
t(\neg\phi) &:= t(\phi) \to \bot, \\
t(\phi \wedge \psi) &:= (t(\psi) \to (t(\psi) \to t(\phi)) \to \bot) \to \bot, \\
t(\phi \vee \psi) &:= (t(\phi) \to t(\psi)) \to t(\psi).
\end{aligned}
$$

$t$ is a truth-preserving and a provability-preserving translation.

### Example: Truth-preserving

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![p \vee q]\!]$ | $[\![t(p \vee q)]\!]$ | $[\![p \wedge q]\!]$ | $[\![t(p \wedge q)]\!]$ | etc. |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 1 | 1 | 1 | 0 | 0 | |
| 1 | 0 | 1 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 1 | 1 | |

# Preserving Translations

## Example: Standard to Minimal Translation

Let $t : \mathcal{F}_S \to \mathcal{F}_M$ be:

$$\begin{aligned} t(\top) &:= \bot \to \bot, \\ t(\neg\phi) &:= t(\phi) \to \bot, \\ t(\phi \wedge \psi) &:= (t(\psi) \to (t(\psi) \to t(\phi)) \to \bot) \to \bot, \\ t(\phi \vee \psi) &:= (t(\phi) \to t(\psi)) \to t(\psi). \end{aligned}$$

$t$ is a truth-preserving and a provability-preserving translation.

## Example: Truth-preserving

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![p \vee q]\!]$ | $[\![t(p \vee q)]\!]$ | $[\![p \wedge q]\!]$ | $[\![t(p \wedge q)]\!]$ | etc. |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 1 | 1 | 1 | 0 | 0 | |
| 1 | 0 | 1 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | 1 | 1 | |

# Preserving Translations

## Example: Provability-preserving

$$\dfrac{\begin{matrix}\mathcal{D}_l & \mathcal{D}_r \\ \phi & \psi\end{matrix}}{\phi \wedge \psi}\ \wedge_i \qquad \Rightarrow \qquad \begin{matrix} \begin{matrix}\mathcal{D}'_l & \mathcal{D}'_r \\ t(\phi) & t(\psi)\end{matrix} \\ \mathcal{D}' \\ t(\phi \wedge \psi)\end{matrix}$$

# Preserving Translations

## Example: Provability-preserving

$$\dfrac{\begin{matrix}\mathcal{D}_l & \mathcal{D}_r \\ \phi & \psi\end{matrix}}{\phi \wedge \psi} \wedge_i \quad \Rightarrow \quad \begin{matrix}\begin{matrix}\mathcal{D}'_l & \mathcal{D}'_r \\ t(\phi) & t(\psi)\end{matrix} \\ \mathcal{D}' \\ t(\phi \wedge \psi)\end{matrix}$$

# Preserving Translations

**Example: Provability-preserving**

$$\dfrac{\begin{matrix}\mathcal{D}_l & \mathcal{D}_r \\ \phi & \psi\end{matrix}}{\phi \wedge \psi}\wedge_i \quad \Rightarrow \quad \begin{matrix}\begin{matrix}\mathcal{D}'_l & \mathcal{D}'_r \\ t(\phi) & t(\psi)\end{matrix} \\ \mathcal{D}' \\ (t(\psi) \to (t(\psi) \to t(\phi)) \to \bot) \to \bot\end{matrix}$$

# Preserving Translations

## Example: Provability-preserving

$$\dfrac{\begin{array}{c}\mathcal{D}_l \\ \phi\end{array} \quad \begin{array}{c}\mathcal{D}_r \\ \psi\end{array}}{\phi \wedge \psi}\ \wedge_i \qquad \Rightarrow \qquad \begin{array}{c} \begin{array}{cc}\mathcal{D}'_l & \mathcal{D}'_r \\ t(\phi) & t(\psi)\end{array} \\ \mathcal{D}' \\ (t(\psi) \to (t(\psi) \to t(\phi)) \to \bot) \to \bot \end{array}$$

# Preserving Translations

## Example: Provability-preserving

$$\dfrac{\begin{matrix}\mathcal{D}_l & \mathcal{D}_r \\ \phi & \psi\end{matrix}}{\phi \wedge \psi}\wedge_i \qquad \Rightarrow \qquad \begin{matrix}\mathcal{D}'_l \quad \mathcal{D}'_r \\ \phi \qquad \psi \\ \mathcal{D}' \\ (\psi \to (\psi \to \phi) \to \bot) \to \bot\end{matrix}$$

# Preserving Translations

## Example: Provability-preserving

$$\dfrac{\begin{array}{cc}\mathcal{D}_l & \mathcal{D}_r \\ \phi & \psi\end{array}}{\phi \wedge \psi} \wedge_i \quad \Rightarrow \quad \dfrac{\begin{array}{c}\begin{array}{cc}\mathcal{D}'_l & \mathcal{D}'_r \\ \phi & \psi\end{array} \\ \mathcal{D}' \\ \bot\end{array}}{(\psi \to (\psi \to \phi) \to \bot) \to \bot} \to_i, u$$

# Preserving Translations

## Example: Provability-preserving

$$\dfrac{\begin{matrix}\mathcal{D}_l \\ \phi\end{matrix} \quad \begin{matrix}\mathcal{D}_r \\ \psi\end{matrix}}{\phi \wedge \psi}\ \wedge_i \qquad \Rightarrow \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}'_r \\ \psi \\ \mathcal{D}'_1 \\ (\psi \to \phi) \to \bot\end{matrix} \quad \begin{matrix}\mathcal{D}'_l \\ \phi \\ \mathcal{D}'_2 \\ \psi \to \phi\end{matrix}}{\bot}\ \to_e}{(\psi \to (\psi \to \phi) \to \bot) \to \bot}\ \to_i, u$$

# Preserving Translations

## Example: Provability-preserving

$$
\frac{\begin{matrix}\mathcal{D}_l \\ \phi\end{matrix} \qquad \begin{matrix}\mathcal{D}_r \\ \psi\end{matrix}}{\phi \wedge \psi}\wedge_i
\qquad \Rightarrow \qquad
\frac{\dfrac{\dfrac{\psi \to (\psi \to \phi) \to \bot^u \qquad \begin{matrix}\mathcal{D}'_r \\ \psi\end{matrix}}{(\psi \to \phi) \to \bot}\to_e \qquad \begin{matrix}\mathcal{D}'_l \\ \phi \\ \mathcal{D}'_2 \\ \psi \to \phi\end{matrix}}{\bot}\to_e}{(\psi \to (\psi \to \phi) \to \bot) \to \bot}\to_i, u
$$

# Preserving Translations

## Example: Provability-preserving

$$
\dfrac{\begin{matrix}\mathcal{D}_l \\ \phi\end{matrix} \qquad \begin{matrix}\mathcal{D}_r \\ \psi\end{matrix}}{\phi \wedge \psi}\,\wedge_i
\qquad \Rightarrow \qquad
\dfrac{\dfrac{\dfrac{\psi \to (\psi \to \phi) \to \bot^{u} \qquad \begin{matrix}\mathcal{D}'_r \\ \psi\end{matrix}}{(\psi \to \phi) \to \bot}\,\to_e \qquad \dfrac{\begin{matrix}\mathcal{D}'_l \\ \phi\end{matrix}}{\psi \to \phi}\,\to_i, v}{\bot}\,\to_e}{(\psi \to (\psi \to \phi) \to \bot) \to \bot}\,\to_i, u
$$

# Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

**Example: duality of formulas**

| tautology | contradiction |
|---|---|
| $\wedge$ | $\vee$ |
| $\vee$ | $\wedge$ |
| $\neg$ | $\neg$ |
| $\rightarrow$ | ? |
| ? | $\rightarrow$ |

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\quad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
- Also for: $[\![\neg(\phi \rightarrow \psi)]\!] = [\![\neg\phi \not\leftarrow \neg\psi]\!]$ $\quad [\![\neg(\phi \not\leftarrow \psi)]\!] = [\![\neg\phi \rightarrow \neg\psi]\!]$

# Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

**Example: duality of formulas**

| tautology | contradiction |
|---|---|
| $\wedge$ | $\vee$ |
| $\vee$ | $\wedge$ |
| $\neg$ | $\neg$ |
| $\to$ | ? |
| ? | $\to$ |

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\quad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
- Also for: $[\![\neg(\phi \to \psi)]\!] = [\![\neg\phi \not\leftarrow \neg\psi]\!]$ $\quad [\![\neg(\phi \not\leftarrow \psi)]\!] = [\![\neg\phi \to \neg\psi]\!]$

# Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

**Example: duality of formulas**

| tautology | contradiction |
|---|---|
| $\wedge$ | $\vee$ |
| $\vee$ | $\wedge$ |
| $\neg$ | $\neg$ |
| $\to$ | ? |
| ? | $\to$ |

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\quad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
- Also for: $[\![\neg(\phi \to \psi)]\!] = [\![\neg\phi \not\leftarrow \neg\psi]\!]$ $\quad [\![\neg(\phi \not\leftarrow \psi)]\!] = [\![\neg\phi \to \neg\psi]\!]$

# Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

## Example: duality of formulas

| tautology | contradiction |
|---|---|
| $\wedge$ | $\vee$ |
| $\vee$ | $\wedge$ |
| $\neg$ | $\neg$ |
| $\to$ | ? |
| ? | $\to$ |

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\qquad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
- Also for: $[\![\neg(\phi \to \psi)]\!] = [\![\neg\phi \not\leftarrow \neg\psi]\!]$ $\qquad [\![\neg(\phi \not\leftarrow \psi)]\!] = [\![\neg\phi \to \neg\psi]\!]$

# Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

## Example: duality of formulas

| tautology | contradiction |
|---|---|
| $\wedge$ | $\vee$ |
| $\vee$ | $\wedge$ |
| $\neg$ | $\neg$ |
| $\rightarrow$ | $\not\leftarrow$ |
| $\not\leftarrow$ | $\rightarrow$ |

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\quad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
- Also for: $[\![\neg(\phi \rightarrow \psi)]\!] = [\![\neg\phi \not\leftarrow \neg\psi]\!]$ $\quad [\![\neg(\phi \not\leftarrow \psi)]\!] = [\![\neg\phi \rightarrow \neg\psi]\!]$

# Outline of Contribution (2)

- We have seen how to preserve truth and provability by translation.
- When is a system sound and complete w.r.t. classical semantics?
- Can we find another such system?

## Example: duality of formulas

| tautology | contradiction |
|---|---|
| $\wedge$ | $\vee$ |
| $\vee$ | $\wedge$ |
| $\neg$ | $\neg$ |
| $\rightarrow$ | $\not\leftarrow$ |
| $\not\leftarrow$ | $\rightarrow$ |

- DeMorgan: $[\![\neg(\phi \wedge \psi)]\!] = [\![\neg\phi \vee \neg\psi]\!]$ $\quad [\![\neg(\phi \vee \psi)]\!] = [\![\neg\phi \wedge \neg\psi]\!]$
- Also for: $[\![\neg(\phi \rightarrow \psi)]\!] = [\![\neg\phi \not\leftarrow \neg\psi]\!]$ $\quad [\![\neg(\phi \not\leftarrow \psi)]\!] = [\![\neg\phi \rightarrow \neg\psi]\!]$

# Arrow Formulas

## Example: Arrow Formulas

Let $\mathcal{F}_A$ be the smallest set such that:

$$a \in \mathcal{F}_A,$$
$$\phi \rightarrow \psi \in \mathcal{F}_A,$$
$$\phi \not\leftarrow \psi \in \mathcal{F}_A,$$

where $\phi, \psi \in \mathcal{F}_A$.

## Example: Classical Arrow Semantics

| $[\![\phi]\!]$ | $[\![\psi]\!]$ | $[\![\phi \rightarrow \psi]\!]$ | $[\![\phi \not\leftarrow \psi]\!]$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 |

**Example: a set of rules (Arrow Natural Deduction)**

Given some $\chi \in \mathcal{F}_A$.

$$\frac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi \to \psi & \phi\end{matrix}}{\psi} \to_{e_1} \qquad \frac{\begin{matrix}\mathcal{D} \\ \neg(\phi \to \psi)\end{matrix}}{\phi} \to_{e_2}$$

$$\frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \neg\phi & \psi\end{matrix}}{\phi \not\leftarrow \psi} \not\leftarrow_i \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\neg\phi} \not\leftarrow_{e_1} \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\psi} \not\leftarrow_{e_2}$$

Where $\neg\phi := \phi \to (\chi \not\leftarrow \chi)$.

# Arrow Formulas

## Example: a set of rules (Arrow Natural Deduction)

Given some $\chi \in \mathcal{F}_A$.

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi \to \psi & \phi\end{matrix}}{\psi} \to_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \neg(\phi \to \psi)\end{matrix}}{\phi} \to_{e_2}$$

$$\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \neg\phi & \psi\end{matrix}}{\phi \not\leftarrow \psi} \not\leftarrow_i \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\neg\phi} \not\leftarrow_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\psi} \not\leftarrow_{e_2}$$

Where $\neg\phi := \phi \to (\chi \not\leftarrow \chi)$.

## Example: a set of rules (Arrow Natural Deduction)

Given some $\chi \in \mathcal{F}_A$.

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi \to \psi & \phi\end{matrix}}{\psi} \to_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \neg(\phi \to \bot)\end{matrix}}{\phi} \to_{e_2}$$

$$\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \neg\phi & \psi\end{matrix}}{\phi \not\leftarrow \psi} \not\leftarrow_i \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\neg\phi} \not\leftarrow_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\psi} \not\leftarrow_{e_2}$$

Where $\neg\phi := \phi \to (\chi \not\leftarrow \chi)$.

# Arrow Formulas

## Example: a set of rules (Arrow Natural Deduction)

Given some $\chi \in \mathcal{F}_A$.

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi} \to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \to \psi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \phi\end{matrix}}{\psi} \to_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \neg\neg\phi\end{matrix}}{\phi} \to_{e_2}$$

$$\dfrac{\begin{matrix}\mathcal{D} \\ \neg\phi\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ \psi\end{matrix}}{\phi \not\leftarrow \psi} \not\leftarrow_i \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\neg\phi} \not\leftarrow_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \not\leftarrow \psi\end{matrix}}{\psi} \not\leftarrow_{e_2}$$

Where $\neg\phi := \phi \to (\chi \not\leftarrow \chi)$.

# Arrow Formulas

## Example: a deduction

$$
\dfrac{\dfrac{\dfrac{\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\,{\not\leftarrow}_{e_1} \qquad \dfrac{\phi \not\leftarrow \phi^u}{\phi}\,{\not\leftarrow}_{e_2}}{\chi \not\leftarrow \chi}\,{\to}_{e_1}}{\neg\psi}\,{\to}_{i} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\,{\not\leftarrow}_{e_1} \qquad \dfrac{\phi \not\leftarrow \phi^u}{\phi}\,{\not\leftarrow}_{e_2}}{\chi \not\leftarrow \chi}\,{\to}_{e_1}}{\neg(\psi \to \psi)}\,{\to}_{i}}{\psi}\,{\to}_{e_2}}{\dfrac{\psi \not\leftarrow \psi}{(\phi \not\leftarrow \phi) \to (\psi \not\leftarrow \psi)}\,{\to}_{i},u}\,{\not\leftarrow}_{i}
$$

- (Extensionality of $\phi \to \phi = \psi \to \psi$, and of $\phi \not\leftarrow \phi = \psi \not\leftarrow \psi$?)

# Arrow Formulas

## Example: a deduction

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1}
      \qquad
      \dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}
    }{\chi \not\leftarrow \chi}\to_{e_1}
  }{\neg\psi}\to_i
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1}
        \qquad
        \dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}
      }{\chi \not\leftarrow \chi}\to_{e_1}
    }{\neg(\psi \to \psi)}\to_i
  }{\psi}\to_{e_2}
}{
  \dfrac{\psi \not\leftarrow \psi}{(\phi \not\leftarrow \phi) \to (\psi \not\leftarrow \psi)}\to_{i}, u
}\not\leftarrow_i
$$

- (Extensionality of $\phi \to \phi = \psi \to \psi$, and of $\phi \not\leftarrow \phi = \psi \not\leftarrow \psi$?)

# Arrow Formulas

## Example: a deduction

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \nleftarrow \phi^u}{\neg\phi}\nleftarrow_{e_1} \quad \dfrac{\phi \nleftarrow \phi^u}{\phi}\nleftarrow_{e_2}}{\chi \nleftarrow \chi}\to_{e_1}}{\neg\psi}\to_i \quad \dfrac{\dfrac{\dfrac{\dfrac{\phi\nleftarrow\phi^u}{\neg\phi}\nleftarrow_{e_1}\quad\dfrac{\phi\nleftarrow\phi^u}{\phi}\nleftarrow_{e_2}}{\chi\nleftarrow\chi}\to_{e_1}}{\neg(\psi\to\psi)}\to_i}{\psi}\to_{e_2}}{\psi\nleftarrow\psi}\nleftarrow_i}{(\phi\nleftarrow\phi)\to(\psi\nleftarrow\psi)}\to_{i},u
$$

- (Extensionality of $\phi \to \phi = \psi \to \psi$, and of $\phi \nleftarrow \phi = \psi \nleftarrow \psi$?)

# Arrow Formulas

## Example: a deduction

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1} \qquad \dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}}{\chi \not\leftarrow \chi}\to_{e_1}}{\neg\psi}\to_i \qquad \dfrac{\dfrac{\dfrac{\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1} \qquad \dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}}{\chi \not\leftarrow \chi}\to_{e_1}}{\neg(\psi \to \psi)}\to_i}{\psi}\to_{e_2}}{\psi \not\leftarrow \psi}\not\leftarrow_i}{(\phi \not\leftarrow \phi) \to (\psi \not\leftarrow \psi)}\to_{i}, u
$$

- (Extensionality of $\phi \to \phi = \psi \to \psi$, and of $\phi \not\leftarrow \phi = \psi \not\leftarrow \psi$?)

# Arrow Formulas

## Example: a deduction

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1}
\quad
\dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}
}{\chi \not\leftarrow \chi}\rightarrow_{e_1}
}{\neg\psi}\rightarrow_{i}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\phi \not\leftarrow \phi^u}{\neg\phi}\not\leftarrow_{e_1}
\quad
\dfrac{\phi \not\leftarrow \phi^u}{\phi}\not\leftarrow_{e_2}
}{\chi \not\leftarrow \chi}\rightarrow_{e_1}
}{\neg(\psi \rightarrow \psi)}\rightarrow_{i}
}{\psi}\rightarrow_{e_2}
}{
\dfrac{\psi \not\leftarrow \psi}{(\phi \not\leftarrow \phi) \rightarrow (\psi \not\leftarrow \psi)}\rightarrow_{i}, u
}\not\leftarrow_{i}
$$

- (Extensionality of $\phi \rightarrow \phi = \psi \rightarrow \psi$, and of $\phi \not\leftarrow \phi = \psi \not\leftarrow \psi$?)

# Summary

- We can show soundness and completeness by truth-preserving and provability-preserving translations.
- We have seen alternative connectives for classical propositional logic.

# Summary

- We can show soundness and completeness
  by truth-preserving and provability-preserving translations.
- We have seen alternative connectives for classical propositional logic.