Alternative Connectives for Classical Propositional Logic

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Abstract

Classical natural deduction systems are related to a similar dual natural deduction system. We introduce a natural deduction system for an alternative set of connectives consisting of implication ($\rightarrow$) and its dual ($\not\leftarrow$). A proof of the soundness and completeness of the alternative natural deduction system with respect to a natural semantics via an interpretation to usual classical propositional logic is outlined.

Subjects Logic in Computer Science, Interpretational Proof Theory

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1 Introduction

Showing the soundness of a natural deduction system is often the easy part of a soundness and completeness proof, since we can check that deduction rules are valid with respect to some semantic interpretation: if all premises are true, is the conclusion of a rule also true? Showing the completeness part is more involved: we need to provide an argument that all true formulas under a true interpretation are derivable. In this work, we explore an alternative approach to soundness and completeness proofs.

Instead of directly proving these properties of a natural deduction system, we use a framework of translating a logical system into another logical system that is already known to be sound and complete. These translations, called interpretations, have two requirements: (1) an interpretation must preserve truth, i.e. a formula represented in both systems must have the same truth value; (2) an interpretation must preserve the provability of formulas in both natural deduction systems, i.e. for all deductions of one natural deduction system, a deduction must be shown to exist in the other natural deduction system and vice versa. This framework is largely based on Grabmayer's abstract natural deduction systems [Gra05].

The motivation for writing this paper is in trying to prove correctness of Java programs. This has lead the author to several interesting papers, e.g. [RB12] [Gen33] [D'A99] [Hun33] [Smu95] [WL11], and an investigation of an alternative natural deduction system began. The paper by Riedel & Bruck was inspirational with regard to duality and partial valuations of propositional formulas. The paper of Wu & Li suggested a deep symmetry in natural deduction systems, namely to employ a dual deduction system for deducing negated formulas. Similar to Wu & Li, the author has found an interest in category theory [ML78] and with regard to propositional logic, a system that operates purely on arrows evolved.

The structure of this paper is as follows: in Section 2 basic preliminaries are discussed. In Section 3, an exposition of propositional logic is given as usual, but the definitions are accommodating towards the introduction of the alternative natural deduction system: a semantic interpretation of formulas; functions that are truth-preserving; (abstract) natural deduction systems; the notion of derivability, and; functions that are provability-preserving. In Section 4, we explore three systems: a system that is known to be sound and complete with respect to classical semantics under certain axiom admissions, namely minimal and intuitionistic propositional logic; an argument for the duality of natural deduction systems and a conjecture connecting duality with classical semantics, and; the introduction of an alternative natural deduction system.

The reader is advised to skip Sections 2 and 3 on a first read, and to only consult those sections if no(ta)tions are unclear.

2 Preliminaries

In this section, we discuss the most important basic no(ta)tions. Not all important notions will be discussed in this section. This work is a study of first-order classical propositional logic without predicates, functions, quantifiers, first-order variables. It is assumed that the reader is familiar with this kind of logic. We base our preliminaries on the work of others and use standard notation for sets, functions and sequences.

The objects we study are symbols or sets of objects or sequences of objects, and are staged at the object level, as written on paper. Our study of these objects or collections of objects takes place on the meta level, in the mind of the reader.

<u>Set Theory</u> By $\varnothing$ we denote the empty set. By convention, italic uppercase roman letter are used as variables for sets. Let a set A be given. We only consider countable sets: finite sets $A = \{\phi_1, \ldots \phi_n\}$ or countably infinite sets $A = \{\phi_1, \ldots\}$, given distinct objects ϕ_i indexed by $1 \leq i\,(\leq n)$. By $|A|$ we denote the cardinality of a set, which is some fixed n for finite sets and

∞ for infinite sets. A set of cardinality 1 is called a singleton. We will use set comprehension notation $\{\phi \mid P(\phi)\}$. By $\mathbb{N}$ we denote the countably infinite set of natural numbers $\{0, 1, \ldots\}$.

Let two sets A and B be given. By the power set $\mathcal{P}(A) \coloneqq \{S \mid S \subseteq A\}$ we denote the set of all subsets of A. The Cartesian product $A \times B \coloneqq \{\langle a, b\rangle \mid a \in A \text{ and } b \in B\}$ has ordered pairs $\langle a, b\rangle$ as elements. We also use extended pairs: triples, quadruples, etc., where generally $\langle a_1, a_2, \ldots, a_n\rangle \in A_1 \times A_2 \times \cdots \times A_n$ is known as the n-tuple for some number $n > 0$. The repeated product $A^n \coloneqq \{\langle a_1, \ldots, a_n\rangle \mid a_i \in A \text{ for } 1 \leq i \leq n\}$, and $A^0 \coloneqq \varnothing$, denotes the set of all n-tuples over the same set A for $n \geq 0$.

We use, in their usual meaning: the membership relation $\in$, the subset relation $\subseteq$, the proper subset relation $\subset$, the union operation $\cup$, the intersection operation $\cap$, and the Kleene star operation $A^* \coloneqq A^0 \cup A^1 \cup \cdots$ for any set A. Operations and relations are not defined as objects, and we merely use them for denotation on the meta level.

Functions Let $f : D \to C$ be a function. It is defined as a subset of $D \times C$, that maps $d \in D$ to $c \in C$ written as $d \mapsto c$. $\operatorname{dom} f \coloneqq D$ is the domain and $\operatorname{codom} f \coloneqq C$ the co-domain. For every element $d \in D$ at most a single mapping $d \mapsto c$ exists, regardless of $c \in C$. If a mapping $d \mapsto c$ exists we say that f is defined for d. By $f(d) = c$ we denote function application. For two functions $g : B \to C$ and $h : A \to B$, the composition $g \circ h : A \to C$ is defined as $g \circ h(a) \coloneqq g(h(a))$. By id we denote the identity function $\mathrm{id}(x) = x$ for any x. Function f is total if f is defined for every $d \in D$, f is partial otherwise. The image of f, denoted as $f^{\to} : \mathcal{P}(D) \to \mathcal{P}(C)$, is defined as the set $f^{\to}(A) \coloneqq \{f(d) \mid d \in D \text{ and } d \in A\}$. The pre-image of f, denoted $f^{\leftarrow} : \mathcal{P}(C) \to \mathcal{P}(D)$, is defined as the set $f^{\leftarrow}(B) \coloneqq \{c \mid c \in C \text{ and } f(c) \in B\}$. The inverse of a function, denoted as $f^{-1} : C \to \mathcal{P}(D)$, is the pre-image of a singleton subset of the co-domain of f. We call f a bijection if every element of $\operatorname{codom} f^{-1}$ is a singleton, and we note the inverse as $f^{-1} : C \to D$.

3 Propositional Logic

We will look at formulas of first-order propositional logic. Formulas are constructed from connectives and atomic, indivisible propositions. By $\mathcal{A} = \{a_1, \ldots\}$ we denote the countably infinite set of atomic propositions[1], so-called propositional variables. Definition 3.1 (on the next page) is defined more abstractly than usual since we will explore different sets of connectives.

Definition 3.1. (*Formulas*) Let C be a set of connectives with $C = \mathrm{Con} \cup \mathrm{Uni} \cup \mathrm{Bin}$ where $\mathrm{Con}, \mathrm{Uni}, \mathrm{Bin}$ are some disjoint sets of nullary, unary, and binary connectives, respectively. Then, the set $\mathcal{F}_C$ of formulas of propositional logic with connectives of from C is defined as the smallest set such that:

$$\begin{aligned} a &\in \mathcal{F}_C && \text{if } a \in \mathcal{A}, \\ \Box &\in \mathcal{F}_C && && \text{if } \Box \in \mathrm{Con}, \\ (\Box\phi) &\in \mathcal{F}_C && \text{if } \phi \in \mathcal{F} && \text{and } \Box \in \mathrm{Uni}, \\ (\phi \Box \psi) &\in \mathcal{F}_C && \text{if } \phi, \psi \in \mathcal{F} && \text{and } \Box \in \mathrm{Bin}. \end{aligned}$$

We assume C is finite. The subscript of $\mathcal{F}_C$ is dropped if C is clear from context or unnecessary. Every connective symbol is member of at most one of the sets $\mathrm{Con}, \mathrm{Uni}, \mathrm{Bin}$.

We use Greek lower-case letters for meta-variables of formulas. Standard connectives have their usual names: $\top$ for top, $\bot$ for bottom, $\neg$ for negation, $\wedge$ for conjunction, $\vee$ for disjunction and $\to$ for implication. We also study a non-standard connective $\not\leftarrow$, which we call not-implied-by.

[1] Some authors regard $\top$ and $\bot$ as atomic propositions [VD83]. Here those connectives are constants, just as in [TS00].

Example 3.2. (*Standard connectives*) The set of standard connectives is $S \coloneqq \{\top, \bot, \neg, \wedge, \vee, \rightarrow\}$ where $\mathrm{Con} = \{\top, \bot\}$, $\mathrm{Uni} = \{\neg\}$, $\mathrm{Bin} = \{\wedge, \vee, \rightarrow\}$. Then, the set of standard formulas is $\mathcal{F}_S$.

Remark 3.3. (*Saving on parentheses*) Formally, elements of $\mathcal{F}_C$ have all their parentheses. We will save on the number of parentheses throughout this work: (1) we never write the outermost pair, (2) if we applied more parentheses than necessary around an otherwise valid formula ϕ, we silently discard unnecessary parenthesis, e.g. $(\phi) = \phi$, and (3) if other parentheses are missing we apply a usual precedence rule in the order: $\neg, \wedge, \vee, \rightarrow, \nleftarrow$, such that $\neg$ binds strongly and $\nleftarrow$ binds weakly. Every binary connective is considered right-associative, i.e. $\phi \square \psi \square \chi = \phi \square (\psi \square \chi)$. These informal rules are convenient and conventional.

An important relation between two formulas is the sub-formula relation. Given two formulas $\phi, \psi \in \mathcal{F}_C$. By the sub-formula relation $\phi \leq \psi$ we denote that the formula ϕ occurs in the construction of ψ. Every formula is a sub-formula of itself, i.e. $\phi \leq \phi$. The proper sub-formula relation $\phi < \psi$ denotes $\phi \leq \psi$ and $\phi \neq \psi$. A direct sub-formula is a proper sub-formula that is directly used in the construction of an outer formula. For example $\psi \vee (\phi \rightarrow \phi)$ the two direct sub-formulas are: ψ and $\phi \rightarrow \phi$. Here ϕ is not a direct sub-formula of the outer formula. However, ϕ is a direct sub-formula of $\phi \rightarrow \phi$.

3.1 Interpretation

The classical interpretation of a formula is one of denotation—the ideal result of a formula (see 1.2 of [VD83] or 1.4.1 of [HR04]). We assume that atomic propositions have a known denotation, given by the primitive valuation $v : \mathcal{A} \rightarrow \mathbb{B}$ where $\mathbb{B} = \{0, 1\}$ is the value set consisting of only two distinct symbols. We extend primitive valuations to a valuation that is a function of formulas in the definition below. Our definition also includes the non-standard connective $\nleftarrow$, which will be extensively studied in Sections 4.2 and 4.3.

Definition 3.4. (*Valuation*) We define the homomorphic *valuation* $[\![\cdot]\!]_{\cdot} : (\mathcal{A} \rightarrow \mathbb{B}) \rightarrow \mathcal{F}_C \rightarrow \mathbb{B}$ with respect to some set $C \subseteq \{\top, \bot, \neg, \wedge, \vee, \rightarrow, \nleftarrow\}$, where we extend every primitive valuation v to a valuation for all formulas $[\![\cdot]\!]_v : \mathcal{F}_C \rightarrow \mathbb{B}$, such that for all $\phi, \psi \in \mathcal{F}_C$:

$$\text{if } [\![\phi]\!]_v = \begin{matrix}0\\0\\1\\1\end{matrix} \text{ and } [\![\psi]\!]_v = \begin{matrix}0\\1\\0\\1\end{matrix} \text{ then } \quad [\![\phi \rightarrow \psi]\!]_v \coloneqq \begin{matrix}1\\1\\0\\1\end{matrix}, [\![\phi \nleftarrow \psi]\!]_v \coloneqq \begin{matrix}0\\1\\0\\0\end{matrix},$$

$$\text{if } [\![\phi]\!]_v = \begin{matrix}0\\0\\1\\1\end{matrix} \text{ and } [\![\psi]\!]_v = \begin{matrix}0\\1\\0\\1\end{matrix} \text{ then } \quad [\![\phi \vee \psi]\!]_v \coloneqq \begin{matrix}0\\1\\1\\1\end{matrix}, [\![\phi \wedge \psi]\!]_v \coloneqq \begin{matrix}0\\0\\0\\1\end{matrix},$$

$$\text{if } [\![\phi]\!]_v = \begin{matrix}0\\1\end{matrix} \text{ then } \quad [\![\neg\phi]\!]_v \coloneqq \begin{matrix}1\\0\end{matrix},$$

$$[\![\top]\!]_v \coloneqq 1, [\![\bot]\!]_v \coloneqq 0, [\![a]\!]_v \coloneqq v(a) \text{ for } a \in \mathcal{A}.$$

Valuations can be seen as a method for computing the value of some formula. If we list all propositional variables occurring in some given formula in a table and, for every row, a unique primitive valuation v is given to the propositional variables and the column corresponding to the formula is computed using the definition above, that column contains all values the given formula can possibly have. Hence, the value of a formula depends only on the primitive valuation. Colloquially this is called the truth-table method.

Definition 3.5. (*Entailment*) Given a finite set of formulas $\Gamma \subset \mathcal{F}$ and a formula $\psi \in \mathcal{F}$. The *entailment* $\Gamma \vDash \psi$ holds if and only if for all primitive valuations v, $[\![\psi]\!]_v = 1$ if $[\![\phi]\!]_v = 1$ for all $\phi \in \Gamma$.

Remark 3.6. (*Single-element notation*) Instead of $\{\phi_1, \ldots, \phi_n\} \vDash \psi$ we drop the set brackets and write $\phi_1, \ldots, \phi_n \vDash \psi$. In particular, for $\varnothing \vDash \psi$ we write $\vDash \psi$.

If two formulas have the same valuation, regardless of primitive valuation, we consider the formulas equivalent[2] as is defined below. An important subset of equivalences are tautologies: formulas equivalent to $\top$. As expected, equivalence is reflexive, transitive and symmetric.

An equivalence class is a set of formulas for which any two elements are equivalent. Equivalence and entailment are closely related since $\phi \equiv \psi$ if and only if $\phi \vDash \psi$ and $\psi \vDash \phi$.

Definition 3.7. (*Equivalence*) Two formulas $\phi, \psi \in \mathcal{F}$ are *equivalent*, $\phi \equiv \psi$, if and only if for all primitive valuations v we have $[\![\phi]\!]_v = [\![\psi]\!]_v$. ϕ is a tautology if and only if $\phi \equiv \top$. ϕ is a contradiction if and only if $\phi \equiv \bot$.

3.2 Truth-preserving

We are interested in mapping formulas constructed with non-standard connectives (or only with a subset of standard connectives) to standard formulas and vice versa. A function of formulas is an interpretation. See the definition below for a particular kind of interpretations: functions that map formulas to formulas while preserving classical semantics. We will use truth-preserving interpretations in the soundness and completeness proofs in Section 4.

Definition 3.8. (*Truth-preserving interpretation*) A function $t : \mathcal{F}_C \to \mathcal{F}_D$ with respect to sets of connectives C, D is a *truth-preserving interpretation* if and only if for every entailment $\Gamma \vDash \psi$ it holds that $t(\Gamma) \vDash t(\psi)$.

By the identity interpretation we mean an interpretation function that is recursive in direct sub-formulas and maps connectives to themselves, i.e. $t(\square) = \square$ for $\square \in \mathrm{Con}$, $t(\square\phi) = \square t(\phi)$ for $\square \in \mathrm{Uni}$ and $t(\phi \,\square\, \psi) = t(\phi) \,\square\, t(\psi)$ for $\square \in \mathrm{Bin}$.

Example 3.9. There is an interpretation called material implication that is an identity interpretation except for the implication connective where $t(\phi \to \psi) = \neg t(\phi) \vee t(\psi)$. Similarly, another interpretation is called intuitive negation and is an identity interpretation except for the negation connective where $t(\neg\phi) = t(\phi) \to \bot$. We can check that these interpretations are truth-preserving by the truth-table method, see Figure 2.

3.3 Natural Deduction

A precise description of natural deduction systems in the form of trees is given similar to [TS00]. For a more basic introduction, see [HR04] pp. 5–29. We assume the reader is familiar with proof trees and their terminology. In this section we define a natural deduction system $\mathbf{Nc}$ with respect to formulas in $\mathcal{F}_S$, viz. standard natural deduction. In later sections we also define natural deduction systems for formulas constructed from other sets of connectives. We introduce the notion of natural

[2] Some authors[HR04] define equivalence in terms of entailment, i.e. $\phi \equiv \psi$ if and only if $\phi \vDash \psi$ and $\psi \vDash \phi$. Another author[VD83] defines equivalence as a connective $\leftrightarrow$.

deduction systems abstractly, that is useful for defining non-standard natural deduction systems. For a formal account of abstract natural deduction systems see Appendix B.2 of [Gra05].

A natural deduction system with respect to a set of formulas $\mathcal{F}_C$ consists of rules and axioms, where *axioms are special rules with no premises*. An instantiation of a rule has for each formula meta-variable an actual substituted element of $\mathcal{F}_C$; an abstract rule is represented by the set of all instantiations of the rule. A rule application (either abstract or instantiated) is a rule where for each deduction meta-variable and marker meta-variable an actual element is substituted. Conclusions of rules may only use connectives in the set C. Applications of instantiated rules are deductions. See Figure 1 for an abstract deduction.

Deductions are represented by proof trees constructed from leafs and nodes where nodes are labeled by some rule. Assumptions are formulas in $\mathcal{F}_C$ and can occur only at the leafs of a proof tree. The number of branches of a node is equal to the number of premises of a rule. We assume a countably infinite set of markers, for which we use the letters u, w and use natural numbers as instantiations. Assumptions may be marked, e.g. ϕ^u, which are used by rules to close assumptions.

Rules are displayed graphically, either instantiated in a deduction or as a prescription of rules of a deduction system. Rules consists zero or a finite number of premises, which are deductions themselves, and each having a formula as conclusion and a finite number of marked formula occurrences, represented by the leafs of a proof tree. Let us consider a meta-notation for rule $\mathcal{R}$:

$$\dfrac{\begin{matrix}[\psi_1]^{u_1} \\ \mathcal{D}_1 \\ \phi_1\end{matrix} \quad \cdots \quad \begin{matrix}[\psi_n]^{u_n} \\ \mathcal{D}_n \\ \phi_n\end{matrix}}{\psi}\ \mathcal{R}, u_1, \ldots, u_n$$

The notation of the premises has the following meaning, for all i: deduction $\mathcal{D}_i$ has ϕ_i as conclusion (note that ϕ_i is part of $\mathcal{D}_i$). Deduction $\mathcal{D}_i$ has a set $[\psi_i]^{u_i}$ consisting of formula occurrences of ψ_i that are marked by u_i. The markers written after the name of the rule denote that the formula occurrences with the same marker are closed by the application of this rule. We drop the set $[\psi_i]^{u_i}$ and marker u_i if the rule does not close open assumptions for that premise, i.e. we let $[\psi_i]^{u_i} := \varnothing$ if the i-th premise does not close assumptions.

What systems are natural deduction systems is subject of debate, see [Pel99] for a history. We will call something a natural deduction system if it is traditionally called so. Otherwise, we have that a natural deduction system has at least one rule which closes at least one assumption.

We say "a proof tree is constructed inductively," to mean that it is defined similar to Definition 3.10 with respect to induction and for brevity will we only show the rules. We denote $\mathbf{D}^{(+\mathcal{R})}$ for a new natural deduction system in which $\mathcal{R}$ is added to an existing natural deduction system $\mathbf{D}$ as a new rule, and similarly $\mathbf{D}^{(-\mathcal{R})}$ for a natural deduction system in which $\mathcal{R}$ is subtracted.

$$\dfrac{\dfrac{\dfrac{}{\phi \vee \neg\phi}\,\text{LEM} \qquad \phi^1 \qquad \dfrac{(\phi \to \psi) \to \phi^0 \qquad \dfrac{\dfrac{\dfrac{\phi^3 \quad \neg\phi^2}{\bot}\,\neg_e}{\psi}\,\bot_e}{\phi \to \psi}\,\to_i, 3}{\phi}\,\to_e}{\phi}\,\vee_e, 1, 2}{((\phi \to \psi) \to \phi) \to \phi}\,\to_i, 0$$

Figure 1: An abstract proof tree of Peirce's Law without open assumptions, for any $\phi, \psi \in \mathcal{F}_S$ that has the conclusion $((\phi \to \psi) \to \phi) \to \phi$.

Definition 3.10. A proof tree of a natural deduction system $\mathbf{Nc}$ with respect to $\mathcal{F}_S$ is constructed inductively: (1) a single formula occurrence $\phi \in \mathcal{F}_S$ with a marker is a single-node proof tree that represents the deduction with conclusion ϕ and a set of open assumptions containing only ϕ; (2) as shown by the rules below, given that $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ are proof trees of $\mathbf{Nc}$, two markers u, w, and $\phi, \psi, \chi \in \mathcal{F}_S$.

$$\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi & \psi\end{matrix}}{\phi \wedge \psi}\,\wedge_i \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \wedge \psi\end{matrix}}{\phi}\,\wedge_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi \wedge \psi\end{matrix}}{\psi}\,\wedge_{e_2} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \phi\end{matrix}}{\phi \vee \psi}\,\vee_{i_1} \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \psi\end{matrix}}{\phi \vee \psi}\,\vee_{i_2} \qquad \dfrac{\begin{matrix} & [\phi]^u & [\psi]^w \\ \mathcal{D} & \mathcal{D}' & \mathcal{D}'' \\ \phi \vee \psi & \chi & \chi\end{matrix}}{\chi}\,\vee_e, u, w$$

$$\dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \to \psi}\,\to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi \to \psi & \phi\end{matrix}}{\psi}\,\to_e \qquad \dfrac{\begin{matrix}\mathcal{D} \\ \bot\end{matrix}}{\phi}\,\bot_e \qquad \dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi & \neg\phi\end{matrix}}{\bot}\,\neg_e \qquad \dfrac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \bot\end{matrix}}{\neg\phi}\,\neg_i, u \qquad \dfrac{}{\phi \vee \neg\phi}\,\text{LEM} \qquad \dfrac{}{\top}\,\top_i$$

Example 3.11. Continuing with Example 3.9, we may leave out the rules $\neg_i$ and $\neg_e$ if we interpret negation as an intuitive negation: then the rule of $\neg_i$ becomes an instance of $\to_i$ and $\neg_e$ and instance of $\to_e$. Indeed, this results in the system $\mathbf{Nc}^{(--\neg_i\neg_e)}$.

3.4 Derivability

Remark that deductions may still have open assumptions: a proof tree with on its leafs formula occurrences that are not closed by any rule in the construction. For example in Figure 1 the upper right proof tree with conclusion $\phi \to \psi$ still has $\neg\phi^2$ as marked formula occurrence that is not closed by any rule.

Definition 3.12. The set open of marked formula occurrences is defined for all proof trees as:

$$\mathrm{open}(\phi^u) := \{\phi^u\},$$

$$\mathrm{open}\left(\dfrac{\begin{matrix}[\psi_1]^{u_1} & & [\psi_n]^{u_n} \\ \mathcal{D}_1 & & \mathcal{D}_n \\ \phi_1 & \dots & \phi_n\end{matrix}}{\psi}\,\mathcal{R}, u_1, \dots, u_n\right) := \bigcup_{i=1}^{n} \mathrm{open}(\mathcal{D}_i) \backslash [\psi_i]^{u_i}.$$

$\llbracket\phi\rrbracket_v$	$\llbracket\psi\rrbracket_v$	$\llbracket\phi\to\psi\rrbracket_v$	$\llbracket\neg\phi\vee\psi\rrbracket_v$	$\llbracket\neg\phi\rrbracket_v$	$\llbracket\phi\to\bot\rrbracket_v$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	0
1	1	1	1	0	0

Figure 2: Truth-table method for checking equivalence of two truth-preserving interpretations: material implication and intuitive negation, respectively.

Marked formula occurrences still have the mark attached to the formula, hence two formulas that are the same with different marks are different elements in the image of open. We also have the set of open assumptions with respect to a proof tree $\mathcal{D}$, viz. $\mathrm{openset}(\mathcal{D}) = \{\phi \mid \phi^u \in \mathrm{open}(\mathcal{D})\}$, which discards all marks of formula occurrences.

Example 3.13. Consider the proof tree (left) and its open marked formula occurrences (right).

$$\dfrac{\dfrac{\psi^1 \qquad \dfrac{\phi^2 \quad \phi^3}{\phi\wedge\phi}\,\wedge_i}{\psi\wedge\phi\wedge\phi}\,\wedge_i}{\psi\to\psi\wedge\phi\wedge\phi}\,\to_i, 1 \qquad\qquad \dfrac{\dfrac{\{\psi^1\} \qquad \dfrac{\{\phi^2\} \quad \{\phi^3\}}{\{\phi^2,\phi^3\}}}{\{\psi^1,\phi^2,\phi^3\}}}{\{\phi^2,\phi^3\}}$$

It has $\{\phi^2,\phi^3\}$ as open marked formula occurrences and has $\{\phi\}$ as set of open assumptions.

Definition 3.14. (*Derivability*) For any deduction system $\mathbf{D}$, a formula $\psi \in \mathcal{F}$ is *derivable* with respect to a finite set of formulas $\Gamma \subset \mathcal{F}$, denoted by $\Gamma \vdash_{\mathbf{D}} \psi$ if and only if a deduction $\mathcal{D}$ of system $\mathbf{D}$ exists with conclusion ψ and a set of open assumptions $\Gamma = \mathrm{openset}(\mathcal{D})$.

Remark 3.15. (*Single-element notation*) Similar to Remark 3.6, instead of $\{\phi_1,\dots,\phi_n\} \vdash \psi$ we drop the set brackets and write $\phi_1,\dots,\phi_n \vdash \psi$. In particular $\varnothing \vdash \psi$ is written as $\vdash \psi$.

Definition 3.16. (*Soundness and completeness*) A deduction system $\mathbf{D}$ is *sound* with respect to entailment if and only if derivability implies entailment, i.e. $\Gamma \vDash \psi \Rightarrow \Gamma \vdash \psi$. Similarly, deduction system $\mathbf{D}$ is *complete* with respect to entailment if and only if entailment implies derivability, i.e. $\Gamma \vdash \psi \Rightarrow \Gamma \vDash \psi$.

Finally, we assume that the reader is familiar with the following theorem.

Theorem 3.17. $\mathbf{Nc}$ *is sound and complete with respect to* $\vDash$*, i.e. for all* $\Gamma \subset \mathcal{F}_S$ *and* $\psi \in \mathcal{F}_S$ *it holds that* $\Gamma \vdash_{\mathbf{Nc}} \psi$ *if and only if* $\Gamma \vDash \psi$.

3.5 Provability-preserving

Similar to Section 3.2, we are interested in mapping formulas constructed with non-standard connectives (or only with a subset of standard connectives) to standard formulas and vice versa. A function of formulas is an interpretation. See the definition below for a particular kind of interpretations: functions that map formulas to formulas while preserving provability. We will use provability-preserving interpretations in the soundness and completeness proofs in Section 4.

Definition 3.18. (*Provability-preserving interpretation*) Given two natural deduction systems $\mathbf{C}$ with respect to $\mathcal{F}_C$ and $\mathbf{D}$ with respect to $\mathcal{F}_D$. A function $t : \mathcal{F}_C \to \mathcal{F}_D$ is a *provability-preserving interpretation* if and only if a derivation of $\mathbf{C}$ implies a derivation of $\mathbf{D}$ for which the function t is applied to every open assumption and the conclusion, i.e. $\Gamma \vdash_{\mathrm{C}} \psi$ implies $t(\Gamma) \vdash_{\mathrm{D}} t(\psi)$.

Example 3.19. Continuing with Example 3.11, we can see that intuitive negation of Example 3.9 is also a provability-preserving interpretation, shown by the derivations below. Since the interpretation is both truth-preserving and provability-preserving, we can extend the soundness property of $\mathbf{Nc}$ to $\mathbf{Nc}^{(-\neg_i\neg_e)}$. We explore this idea in the next section!

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}' \\ \phi & \neg\phi\end{array}}{\bot}\,\neg_e \quad\Rightarrow\quad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}' \\ \phi\to\bot & \phi\end{array}}{\bot}\,\to_e \qquad\qquad \dfrac{\begin{array}{c}[\phi]^u \\ \mathcal{D} \\ \bot\end{array}}{\neg\phi}\,\neg_i, u \quad\Rightarrow\quad \dfrac{\begin{array}{c}[\phi]^u \\ \mathcal{D} \\ \bot\end{array}}{\phi\to\bot}\,\to_i, u$$

Example 3.20. Post has shown that for $\mathcal{F}_{\{\neg,\vee\}}$ a sound and complete deduction system exists [Pos21]. In his system, $\phi \wedge \psi$ is defined as $\neg(\neg\phi \vee \neg\psi)$, and $\phi \to \psi$ is defined as $\neg\phi \vee \psi$. However, his system could not be considered a natural deduction system, since it had no closing of open assumptions.

4 Alternative Natural Deduction

In this section we explore the interpretation of rules of one natural deduction systems as derivations of another natural deduction system. We work towards the soundness and completeness with respect to entailment by showing a mechanical transformation of proof trees. We first introduce the notion of similarity, $\mathbf{C} \simeq \mathbf{D}$ as an equivalence relation between two natural deduction systems from which we can prove the soundness and completeness of one by the other and vice versa. We also introduce the more strict notion of natural deduction system equivalence, $\mathbf{C} \sim \mathbf{D}$, as a subset of the similarity relation $\mathbf{C} \simeq \mathbf{D}$. Then, in the following three subsections we show:

1. minimal formulas $\mathcal{F}_M$, the minimization interpretation $\cdot^{\dagger}$, two natural deduction systems $\mathbf{Ni}$ and $\mathbf{Nm}$ and two axioms PL and DN, and $\mathbf{Nc} \simeq \mathbf{Ni}^{(+\mathrm{WPL})} \sim \mathbf{Nm}^{(+\mathrm{DN})}$;

2. extended formulas $\mathcal{F}_E$, the dual interpretation $\cdot^d$, a dual natural deduction system $\mathbf{Nc}^d$, the similarity $\mathbf{Nc} \simeq \mathbf{Nc}^d$ and a conjecture with respect to classical semantics;

3. arrow formulas $\mathcal{F}_A$ consisting of only the connectives $\to, \not\leftarrow$, an alternative natural deduction system $\mathbf{Na}$, and $\mathbf{Na} \simeq \mathbf{Nm}^{(+\mathrm{DN})}$ and, therefore, the soundness and completeness of $\mathbf{Na}$ with respect to classical semantics by $\mathbf{Na} \simeq \mathbf{Nm}^{(+\mathrm{DN})} \sim \mathbf{Ni}^{(+\mathrm{WPL})} \simeq \mathbf{Nc}$.

We recall that truth-preserving interpretations are functions $t : \mathcal{F}_C \to \mathcal{F}_D$ for which $\Gamma \vDash \psi$ implies $t(\Gamma) \vDash t(\psi)$, see Definition 3.8. We also recall that provability-preserving interpretations, with respect to two natural deduction systems $\mathbf{C}$ on $\mathcal{F}_C$ and $\mathbf{D}$ on $\mathcal{F}_D$, are functions $t : \mathcal{F}_C \to \mathcal{F}_D$ for which $\Gamma \vdash_{\mathrm{C}} \psi$ implies $t(\Gamma) \vdash_{\mathrm{D}} t(\psi)$, see Definition 3.18.

Similarity and equivalence of natural deduction systems are with respect to, in the general case, two interpretations t_1 and t_2. Interpretations are used in similarity as mapping formulas between two natural deduction systems, as defined below.

Definition 4.1. (*Similarity*) Let two natural deduction systems $\mathbf{C}$, with respect to $\mathcal{F}_C$, and $\mathbf{D}$, with respect to $\mathcal{F}_D$, be given. $\mathbf{C}$ and $\mathbf{D}$ are *similar*, $\mathbf{C} \simeq \mathbf{D}$, with respect to two interpretations $s : \mathcal{F}_C \to \mathcal{F}_D$ and $t : \mathcal{F}_D \to \mathcal{F}_C$ if and only if for all $\Gamma \subseteq \mathcal{F}_C$ and $\psi \in \mathcal{F}_C$ the derivation $\Gamma \vdash_{\mathrm{C}} \psi$ implies $s(\Gamma) \vdash_{\mathrm{D}} s(\psi)$ and for all $\Gamma \subseteq \mathcal{F}_D$ and $\psi \in \mathcal{F}_D$ the derivation $\Gamma \vdash_{\mathrm{D}} \psi$ implies $t(\Gamma) \vdash_{\mathrm{C}} t(\psi)$.

We define *equivalence* as strict similarity, in which only one interpretation is given that is a bijection, and the other is the inverse, i.e. $t = s^{-1}$. A typical bijection is the identity function $\mathrm{id} = \mathrm{id}^{-1}$.

We also outline the notions of rule admissibility[3] and rule derivability. These notions are useful, as we make extensive use of rule derivability in the following subsections. We say that a rule $\mathcal{R}$ is admissible in some natural deduction system $\mathbf{D}$ if and only if $\mathbf{D}$ is equivalent with respect to id to $\mathbf{D}^{(+\mathcal{R})}$. In contrast, a rule $\mathcal{R}$ is derivable in $\mathbf{D}$ if and only if there exists an abstract proof tree in $\mathbf{D}$ with as open assumptions the premises of $\mathcal{R}$ and as conclusion the conclusion of $\mathcal{R}$, and also mimics the closing of assumptions of $\mathcal{R}$. Indeed, whenever a rule $\mathcal{R}$ is derivable in $\mathbf{D}$, it also means that $\mathcal{R}$ is admissible in $\mathbf{D}$, but the converse need not hold, see Lemma B.2.24 in [Gra05].

Finally, we also consider the interpretation of rules of a natural deduction system. Given some interpretation t and natural deduction system $\mathbf{D}$, the natural deduction system $\mathbf{D}^t$ has and only has, for every rule of $\mathbf{D}$, a rule where t is applied to the formulas of premises and assumptions and the conclusion.

4.1 Minimal

In this section we consider formulas $\mathcal{F}_M$ constructed by the set of connectives $M = \{\to, \bot\}$, which we call minimal formulas. We define two *interpretations*, and show they are truth-preserving, since Definition 3.4 also applies for $\mathcal{F}_M$. We define two *natural deduction systems* $\mathbf{Nm}$ and $\mathbf{Ni}$ together with two axioms DN and WPL. We finally show the *equivalence* $\mathbf{Ni}^{(+\mathrm{WPL})} \sim \mathbf{Nm}^{(+\mathrm{DN})}$ and that the interpretations are provability-preserving to show the *similarity* $\mathbf{Nc} \simeq \mathbf{Ni}^{(+\mathrm{WPL})}$.

Interpretations One the two interpretations is given in the definition below. For an intuitive explanation of this interpretation, refer to Section 4.3. Since minimal formulas are a subset of standard formulas, we let the other interpretation be id.

Definition 4.2. For any formula ϕ we inductively construct a formula $\phi^\dagger$ by $\cdot^\dagger : \mathcal{F}_C \to \mathcal{F}_M$, with respect to some $C \subseteq \{\top, \bot, \neg, \wedge, \vee, \to, \not\leftarrow\}$:

$$
\begin{array}{lll}
a^\dagger := a \text{ for all } a \in \mathcal{A}, & (\neg\phi)^\dagger := \phi^\dagger \to \bot & \text{for all } \phi \in \mathcal{F}_C, \\
\top^\dagger := \bot \to \bot, & (\phi \wedge \psi)^\dagger := (\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot & \text{for all } \phi, \psi \in \mathcal{F}_C, \\
\bot^\dagger := \bot, & (\phi \vee \psi)^\dagger := (\phi^\dagger \to \psi^\dagger) \to \psi^\dagger & \text{for all } \phi, \psi \in \mathcal{F}_C, \\
 & (\phi \to \psi)^\dagger := \phi^\dagger \to \psi^\dagger & \text{for all } \phi, \psi \in \mathcal{F}_C, \\
 & (\phi \not\leftarrow \psi)^\dagger := (\psi^\dagger \to \phi^\dagger) \to \bot & \text{for all } \phi, \psi \in \mathcal{F}_C.
\end{array}
$$

The interpretation of $\neg\phi \vee \phi \in \mathcal{F}_S$ is interesting because it is interpreted as $((\phi \to \bot) \to \phi) \to \phi \in \mathcal{F}_M$, called Weak Peirce's Law in [AH03], an instance of Peirce's Law if $\psi = \bot$.

[3] We assume rule admissibility here is admissibility with respect to the consequence relation in terms of Grabmayer's thesis.

Proposition 4.3. *Given $\phi \in \mathcal{F}_S$, it holds that $[\![\phi]\!]_v = [\![\phi^\dagger]\!]_v$ for all primitive valuations v.*

Proof. Checked by the truth-table method. We show by induction that the interpretation is truth-preserving: the base-cases are trivial and induction hypothesis is $[\![\phi]\!]_v = [\![\phi^\dagger]\!]_v$ and $[\![\psi]\!]_v = [\![\psi^\dagger]\!]_v$.

$[\![\phi]\!]_v$	$[\![\psi]\!]_v$	$[\![\neg\phi]\!]_v$	$[\![\phi^\dagger \to \bot]\!]_v$	$[\![\phi \vee \psi]\!]_v$	$[\![(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger]\!]$	etc.
0	0	1	1	0	0	
0	1	1	1	1	1	
1	0	0	0	1	1	
1	1	0	0	1	1	

□

Natural Deduction Systems Below, we define two natural deduction systems, $\mathbf{Nm}$ and $\mathbf{Ni}$, for minimal logic and intuitionistic logic, respectively. These are not equivalent since the rule $\bot_e$ of $\mathbf{Ni}$ is not derivable in $\mathbf{Nm}$.

Definition 4.4. A proof tree of $\mathbf{Nm}$ with respect to $\mathcal{F}_M$ is constructed inductively with the rules:

$$\dfrac{\begin{array}{c}[\phi]^u\\ \mathcal{D}\\ \psi\end{array}}{\phi \to \psi} \to_i, u \qquad\qquad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \phi \to \psi & \phi\end{array}}{\psi} \to_e$$

Remark 4.5. $\mathbf{Nm}$ is actually defined with respect to $\mathcal{F}_{M\setminus\{\bot\}}$. But, as we will see, we need $\bot$ for the axiom DN.

Definition 4.6. A proof tree of $\mathbf{Ni}$ with respect to $\mathcal{F}_M$ is constructed inductively with the rules:

$$\dfrac{\begin{array}{c}[\phi]^u\\ \mathcal{D}\\ \psi\end{array}}{\phi \to \psi} \to_i, u \qquad\qquad \dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{D}'\\ \phi \to \psi & \phi\end{array}}{\psi} \to_e \qquad\qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\phi} \bot_e$$

Definition 4.7. We define the following axioms:

$$\dfrac{}{((\phi \to \bot) \to \bot) \to \phi} \text{ DN} \qquad\qquad \dfrac{}{((\phi \to \bot) \to \phi) \to \phi} \text{ WPL}$$

Remark 4.8. In $\mathbf{Nm}$ it does not hold that Peirce's Law is derivable from WPL [AH03], but in $\mathbf{Ni}$ it does hold, i.e. $\mathbf{Ni}^{(+\text{WPL})} \sim \mathbf{Ni}^{(+\text{PL})}$ for some axiom PL with as conclusion Peirce's Law. We leave this as an exercise for the reader.

Equivalence We show that the equivalence $\mathbf{Ni}^{(+\text{WPL})} \sim \mathbf{Nm}^{(+\text{DN})}$ holds. In all following proofs we assume that there is an unlimited supply of unused distinct markers $u_1, u_2, \ldots$.

Lemma 4.9. $\mathbf{Nm}^{(+\text{DN})} \sim \mathbf{Ni}^{(+\text{WPL})}$

Proof. We show that id is a provability-preserving interpretation from $\mathbf{Nm}^{(+\text{DN})}$ to $\mathbf{Ni}^{(+\text{WPL})}$ and vice versa. (⇒) Given a derivation $\Gamma \vdash \psi$ of $\mathbf{Nm}^{(+\text{DN})}$, we show a derivation $\Gamma \vdash \psi$ of $\mathbf{Ni}^{(+\text{WPL})}$, by showing that all rules of the former are derivable in the latter.

Rules $\to_i$ and $\to_e$ are trivial. The axiom DN of $\mathbf{Nm}^{(+\mathrm{DN})}$ is derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$:

$$\dfrac{\dfrac{\dfrac{}{((\phi\to\bot)\to\phi)\to\phi}\;\mathrm{WPL} \qquad \dfrac{\dfrac{\dfrac{(\phi\to\bot)\to\bot^{u_1} \quad \phi\to\bot^{u_2}}{\bot}\;\to_e}{\phi}\;\bot_e}{(\phi\to\bot)\to\phi}\;\to_i, u_2}{\phi}\;\to_e}{((\phi\to\bot)\to\bot)\to\phi}\;\to_i, u_1$$

We verify that the deduction has no open assumptions.

($\Leftarrow$) Given a derivation $\Gamma \vdash \psi$ of $\mathbf{Ni}^{(+\mathrm{WPL})}$, we show a derivation $\Gamma \vdash \psi$ of $\mathbf{Nm}^{(+\mathrm{DN})}$, by showing that all rules of the former are derivable in the latter.

Rules $\to_i$ and $\to_e$ are again trivial. The rule $\bot_e$of $\mathbf{Ni}^{(+\mathrm{WPL})}$ is derivable in $\mathbf{Nm}^{(+\mathrm{DN})}$:

$$\dfrac{\dfrac{}{((\phi\to\bot)\to\bot)\to\phi}\;\mathrm{DN} \qquad \dfrac{\bot^{u_1}}{(\phi\to\bot)\to\bot}\;\to_i}{\phi}\;\to_e$$

The deduction has one open assumption: $\bot$. Note that the marker of the upper right $\to_i$ is not used, since we do not close any assumptions there.

The axiom WPL of $\mathbf{Ni}^{(+\mathrm{WPL})}$ is derivable in $\mathbf{Nm}^{(+\mathrm{DN})}$. We abbreviate the formula $\mathrm{WPL} := ((\phi\to\bot)\to\phi)\to\phi$ in the following:

$$\dfrac{\dfrac{}{((\mathrm{WPL}\to\bot)\to\bot)\to\mathrm{WPL}}\;\mathrm{DN} \qquad \dfrac{\dfrac{\mathrm{WPL}\to\bot^{u_1} \quad \dfrac{\dfrac{(\phi\to\bot)\to\phi^{u_2} \quad \dfrac{\dfrac{\mathrm{WPL}\to\bot^{u_1} \quad \dfrac{\phi^{u_3}}{\mathrm{WPL}}\;\to_i, u_4}{\bot}\;\to_e}{\phi\to\bot}\;\to_i, u_3}{\phi}\;\to_e}{\mathrm{WPL}}\;\to_i, u_2}{\bot}\;\to_e}{(\mathrm{WPL}\to\bot)\to\bot}\;\to_i, u_1}{((\phi\to\bot)\to\phi)\to\phi}\;\to_e$$

We verify that the deduction has no open assumptions. □

Similarity We finally show that $\mathbf{Ni}^{(+\mathrm{WPL})}$ is similar to $\mathbf{Nc}$. We show that the interpretations $\cdot^\dagger$ and id are provability-preserving. We show that the rules of $\mathbf{Nc}^\dagger$ are derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$ and we show that the rules of $\mathbf{Ni}^{(+\mathrm{WPL})}$ are derivable in $\mathbf{Nc}$.

Lemma 4.10. $\mathbf{Nc} \simeq \mathbf{Ni}^{(+\mathrm{WPL})}$

Proof. ($\Rightarrow$) We show that $\cdot^\dagger$ is a provability-preserving interpretation from $\mathbf{Nc}$ to $\mathbf{Ni}^{(+\mathrm{WPL})}$. Given a derivation $\Gamma \vdash \psi$ of $\mathbf{Nc}$, we show a derivation $\Gamma^\dagger \vdash \psi^\dagger$ of $\mathbf{Ni}^{(+\mathrm{WPL})}$, by showing that all rules of the former are derivable in the latter, i.e. we show only one case of the similarity $\mathbf{Nc}^\dagger \sim \mathbf{Ni}^{(+\mathrm{WPL})}$, where, given a derivation $\Gamma \vdash \psi$ of $\mathbf{Nc}^\dagger$, we show a derivation $\Gamma \vdash \psi$ of $\mathbf{Ni}^{(+\mathrm{WPL})}$.

For the rule $\wedge_i$ of $\mathbf{Nc}$, the interpretation $\wedge_i^\dagger$ of $\mathbf{Nc}^\dagger$ with conclusion $(\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot$ and assumptions $\phi^\dagger$ and $\psi^\dagger$ is derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$.

$$\dfrac{\dfrac{\dfrac{\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot^{u_1} \quad (\psi^\dagger)^{u_2}}{(\psi^\dagger \to \phi^\dagger) \to \bot}\,\to_e \qquad \dfrac{(\phi^\dagger)^{u_3}}{\psi^\dagger \to \phi^\dagger}\,\to_i}{\bot}\,\to_e}{(\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot}\,\to_i, u_1$$

The deduction has two open assumptions: $\phi^\dagger$ and $\psi^\dagger$. Again, the marker of the upper right $\to_i$ is not used. From now on, we no longer note when a marker is left unused.

For the rule $\wedge_{e_1}$ of $\mathbf{Nc}$, the interpretation $\wedge_{e_1}^\dagger$ of $\mathbf{Nc}^\dagger$ with conclusion $\phi^\dagger$ and assumption $(\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot$ is derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$.

$$\dfrac{\dfrac{}{((\phi^\dagger \to \bot) \to \phi^\dagger) \to \phi^\dagger}\,\mathrm{WPL} \qquad \dfrac{\dfrac{\dfrac{((\phi \wedge \psi)^\dagger)^{u_4} \qquad \dfrac{\dfrac{\dfrac{\phi^\dagger \to \bot^{u_1} \qquad \dfrac{(\psi^\dagger \to \phi^\dagger)^{u_3} \quad (\psi^\dagger)^{u_2}}{\phi^\dagger}\,\to_e}{\bot}\,\to_e}{(\psi^\dagger \to \phi^\dagger) \to \bot}\,\to_i, u_3}{\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot}\,\to_i, u_2}{\bot}\,\to_e}{\phi^\dagger}\,\bot_e}{(\phi^\dagger \to \bot) \to \phi^\dagger}\,\to_i, u_1}{\phi^\dagger}\,\to_e$$

The deduction has one open assumption: $(\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot$.

For the rule $\wedge_{e_2}$ of $\mathbf{Nc}$, the interpretation $\wedge_{e_2}^\dagger$ of $\mathbf{Nc}^\dagger$ with conclusion $\psi^\dagger$ and assumption $(\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot$ is derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$.

$$\dfrac{\dfrac{}{((\psi^\dagger \to \bot) \to \psi^\dagger) \to \psi^\dagger}\,\mathrm{WPL} \qquad \dfrac{\dfrac{\dfrac{((\phi \wedge \psi)^\dagger)^{u_3} \qquad \dfrac{\dfrac{\dfrac{(\psi^\dagger)^{u_2} \quad \psi^\dagger \to \bot^{u_1}}{\bot}\,\to_e}{(\psi^\dagger \to \phi^\dagger) \to \bot}\,\to_i}{\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot}\,\to_i, u_2}{\bot}\,\to_e}{\psi^\dagger}\,\bot_e}{(\psi^\dagger \to \bot) \to \psi^\dagger}\,\to_i, u_1}{\psi^\dagger}\,\to_e$$

The deduction has one open assumption: $(\psi^\dagger \to (\psi^\dagger \to \phi^\dagger) \to \bot) \to \bot$.

For the rule $\vee_{i_1}$ of $\mathbf{Nc}$, the interpretation $\vee_{i_1}^\dagger$ of $\mathbf{Nc}^\dagger$ with conclusion $(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger$ and assumption $\phi^\dagger$ is derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$ (below, left). For the rule $\vee_{i_2}$ of $\mathbf{Nc}$, the interpretation $\vee_{i_2}^\dagger$ with conclusion $(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger$ and assumption $\psi^\dagger$ is derivable in $\mathbf{Ni}^{(+\mathrm{WPL})}$ (below, right).

$$\dfrac{\dfrac{(\phi^\dagger \to \psi^\dagger)^{u_1} \quad (\phi^\dagger)^{u_2}}{\psi^\dagger}\, \to_e}{(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger}\, \to_i, u_1 \qquad\qquad \dfrac{(\psi^\dagger)^{u_1}}{(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger}\, \to_i$$

The deductions have as open assumptions: $\phi^\dagger$ (left) and $\psi^\dagger$ (right).

For the rule $\vee_e$ of **Nc**, the interpretation $\vee_e^\dagger$ of $\mathbf{Nc}^\dagger$ with conclusion $\chi^\dagger$ and assumptions $(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger$ and $\phi^\dagger \to \chi^\dagger$ and $\psi^\dagger \to \chi^\dagger$. We show derivability of this rule in $\mathbf{Ni}^{(+\mathrm{WPL})}$ in two intermediate steps:

- $\mathcal{D}_{(1)}$ is a deduction with conclusion $\phi^\dagger \to \bot$ and open assumption $\chi^\dagger \to \bot$:

$$\dfrac{\dfrac{\chi^\dagger \to \bot^{u_1} \qquad \dfrac{\dfrac{\begin{matrix}[\phi^\dagger]^u \\ \mathcal{D}' \\ \chi^\dagger\end{matrix}}{\phi^\dagger \to \chi^\dagger}\, \to_i, u \qquad (\phi^\dagger)^{u_2}}{\chi}\, \to_e}{\bot}\, \to_e}{\phi^\dagger \to \bot}\, \to_i, u_2$$

- $\mathcal{D}_{(2)}$ is a deduction with conclusion $\psi^\dagger \to \bot$ and open assumptions $\chi^\dagger \to \bot$:

$$\dfrac{\dfrac{\chi^\dagger \to \bot^{u_1} \qquad \dfrac{\dfrac{\begin{matrix}[\psi^\dagger]^u \\ \mathcal{D}'' \\ \chi^\dagger\end{matrix}}{\psi^\dagger \to \chi^\dagger}\, \to_i, u \qquad (\psi^\dagger)^{u_3}}{\chi}\, \to_e}{\bot}\, \to_e}{\psi^\dagger \to \bot}\, \to_i, u_3$$

Combined these two deductions make a larger deduction:

$$\dfrac{\dfrac{}{((\chi^\dagger \to \bot) \to \chi^\dagger) \to \chi^\dagger}\, \mathrm{WPL} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{(2)} \\ \psi^\dagger \to \bot\end{matrix} \qquad \dfrac{((\phi^\dagger \to \psi^\dagger) \to \psi^\dagger)^{u_5} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{(1)} \\ \phi^\dagger \to \bot\end{matrix} \quad (\phi^\dagger)^{u_4}}{\bot}\, \to_e}{\psi^\dagger}\, \bot_e}{\phi^\dagger \to \psi^\dagger}\, \to_i, u_4}{\psi}\, \to_e}{\bot}\, \to_e}{\chi^\dagger}\, \bot_e}{(\chi^\dagger \to \bot) \to \chi^\dagger}\, \to_i, u_1}{\chi^\dagger}\, \to_e$$

The deduction has three open assumptions: $(\phi^\dagger \to \psi^\dagger) \to \psi^\dagger$ and $\phi^\dagger \to \chi^\dagger$ and $\psi^\dagger \to \chi^\dagger$.

Rules $\to_i$, $\to_e$, $\bot_e$ are trivial. Rules $\neg_e$ and $\neg_i$ are shown in Example 3.19. We finally show the derivation of the two axioms of $\mathbf{Nc}^\dagger$ in $\mathbf{Ni}^{(+\mathrm{WPL})}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi^\dagger\to\phi^\dagger\to\bot^{u_1}\quad(\phi^\dagger)^{u_2}}{\phi^\dagger\to\bot}\to_e \quad (\phi^\dagger)^{u_2}}{\bot}\to_e}{\phi^\dagger\to\bot}\to_i,u_2}{(\phi^\dagger\to\phi^\dagger\to\bot)\to\phi^\dagger\to\bot}\to_i,u_1 \qquad\qquad \dfrac{\bot^{u_1}}{\bot\to\bot}\to_i,u_1$$

These deductions have no open assumptions.

($\Leftarrow$) Given a derivation $\Gamma\vdash\psi$ of $\mathbf{Ni}^{(+\mathrm{WPL})}$, we show a derivation $\Gamma\vdash\psi$ of $\mathbf{Nc}$, by showing that all rules of the former are derivable in the latter. Rules $\to_i$, $\to_e$ and $\bot_e$ are trivial. The axiom WPL of $\mathbf{Ni}^{(+\mathrm{WPL})}$ is derivable in $\mathbf{Nc}$:

$$\dfrac{\dfrac{\dfrac{}{\phi\vee\neg\phi}\,\mathrm{LEM}\quad \phi^{u_2}\quad \dfrac{(\phi\to\bot)\to\phi^{u_1}\quad \dfrac{\dfrac{\phi^{u_4}\quad\neg\phi^{u_3}}{\bot}\neg_e}{\phi\to\bot}\to_i,u_4}{\phi}\to_e}{\phi}\vee_e,u_2,u_3}{((\phi\to\bot)\to\phi)\to\phi}\to_i,u_1$$

The deduction has no open assumptions. □

Since we have obtained the equivalence $\mathbf{Ni}^{(+\mathrm{WPL})}\sim\mathbf{Nm}^{(+\mathrm{DN})}$ previously, the similarity result also applies to $\mathbf{Nm}^{(+\mathrm{DN})}$, i.e. $\mathbf{Nc}\simeq\mathbf{Nm}^{(+\mathrm{DN})}$, since interpretations can be composed as functions.

4.2 Duality

In this section we consider a super set of standard formulas, the extended formulas $\mathcal{F}_E$, for which we let the set of connectives $E=\{\top,\bot,\neg,\wedge,\vee,\to,\not\leftarrow\}$. We first look at a dual *interpretation*. Then we follow with a dual *natural deduction system*. The dual interpretation is not truth-preserving, however the dual natural deduction system is still similar to a standard natural deduction system. We conjecture that there is a relationship between dual natural deduction systems and classical interpretation.

Interpretation An interpretation of a formula is its dual formula. We define it below as usual; except usually implication has no dual. For this purpose we introduce a connective $\not\leftarrow$, which we call not-implied-by.

The interpretation $\cdot^d$ is not truth-preserving, since for example $[\![\top]\!]_v\neq[\![\top^d]\!]_v$ for all primitive valuations v. However, this interpretation is an involution, i.e. $(\phi^d)^d=\phi$ for every $\phi\in\mathcal{F}_E$.

Definition 4.11. For any formula ϕ we inductively construct a formula ϕ^d by $\cdot^d : \mathcal{F}_C \to \mathcal{F}_E$, with respect to some $C \subseteq \{\top, \bot, \neg, \wedge, \vee, \to, \not\leftarrow\}$:

$$a^d := a \text{ for all } a \in \mathcal{A}, \quad \top^d := \bot, \quad \bot^d := \top, \quad (\neg\phi)^d := \neg(\phi^d) \text{ for all } \phi \in \mathcal{F}_C,$$
$$(\phi \vee \psi)^d := \phi^d \wedge \psi^d \qquad \text{for all } \phi, \psi \in \mathcal{F}_C,$$
$$(\phi \wedge \psi)^d := \phi^d \vee \psi^d \qquad \text{for all } \phi, \psi \in \mathcal{F}_C,$$
$$(\phi \to \psi)^d := \phi^d \not\leftarrow \psi^d \qquad \text{for all } \phi, \psi \in \mathcal{F}_C,$$
$$(\phi \not\leftarrow \psi)^d := \phi^d \to \psi^d \qquad \text{for all } \phi, \psi \in \mathcal{F}_C.$$

Not-implied-by is defined as the dual operation of implication. The laws of De Morgan also apply similarly to implication and not-implied-by. For all primitive valuations v:

$$[\![\neg(\phi \wedge \psi)]\!]_v = [\![\neg\phi \vee \neg\psi]\!]_v, \qquad [\![(\phi \to \psi) \vee (\psi \not\leftarrow \phi)]\!]_v = 1,$$
$$[\![\neg(\phi \vee \psi)]\!]_v = [\![\neg\phi \wedge \neg\psi]\!]_v, \qquad [\![(\phi \not\leftarrow \psi) \wedge (\psi \to \phi)]\!]_v = 0,$$
$$[\![\neg(\phi \to \psi)]\!]_v = [\![\neg\phi \not\leftarrow \neg\psi]\!]_v,$$
$$[\![\neg(\phi \not\leftarrow \psi)]\!]_v = [\![\neg\phi \to \neg\psi]\!]_v.$$

Proposition 4.12. *For every tautology ϕ, the dual ϕ^d is a contradiction, and vice versa.*

Proof. Let $r \circ v : \mathcal{A} \to \mathbb{B}$ be a primitive valuation where every primitive is negated, i.e. $r \circ v(a) = r(v(a))$ where $r(0) = 1$ and $r(1) = 0$. r is an involution, i.e. $r(r(b) = b$ for all $b \in \mathbb{B}$. We verify that $[\![\phi]\!]_v = r([\![\phi^d]\!]_{r \circ v})$, by induction on the structure of ϕ:

- Let $\phi \in \mathcal{A}$, and $[\![a]\!]_v = v(a)$. Then $r([\![a]\!]_{r\circ v}) = r(r \circ v(a)) = v(a)$.
- Let $\phi = \top$, and $[\![\top]\!]_v = 1$. Then $r([\![\bot]\!]_{r\circ v}) = r(0) = 1$. Similar for $\phi = \bot$.
- Let $\phi = \neg\phi_1$, and $[\![\neg\phi_1]\!]_v = r([\![\phi_1]\!]_v)$. Our induction hypothesis is $[\![\phi_1]\!]_v = r([\![\phi_1^d]\!]_{r\circ v})$. Then $r([\![\neg\phi_1^d]\!]_{r\circ v}) = r(r([\![\phi_1^d]\!]_{r\circ v})) = r([\![\phi_1]\!]_v)$.
- Let $\phi = \phi_1 \,\square\, \phi_2$. Our induction hypotheses are $[\![\phi_1]\!]_v = r([\![\phi_1^d]\!]_{r\circ v})$ and $[\![\phi_2]\!]_v = r([\![\phi_2^d]\!]_{r\circ v})$. See the following truth-tables.

$[\![\phi_1]\!]_v$	$[\![\phi_1^d]\!]_{r\circ v}$	$[\![\phi_2]\!]_v$	$[\![\phi_2^d]\!]_{r\circ v}$	$[\![\phi_1 \vee \phi_2]\!]_v$	$[\![\phi_1^d \wedge \phi_2^d]\!]_{r\circ v}$	$[\![\phi_1 \wedge \phi_2]\!]_v$	$[\![\phi_1^d \vee \phi_2^d]\!]_{r\circ v}$
0	1	0	1	0	1	0	1
0	1	1	0	1	0	0	1
1	0	0	1	1	0	0	1
1	0	1	0	1	0	1	0

$[\![\phi_1]\!]_v$	$[\![\phi_1^d]\!]_{r\circ v}$	$[\![\phi_2]\!]_v$	$[\![\phi_2^d]\!]_{r\circ v}$	$[\![\phi_1 \to \phi_2]\!]_v$	$[\![\phi_1^d \not\leftarrow \phi_2^d]\!]_{r\circ v}$	$[\![\phi_1 \not\leftarrow \phi_2]\!]_v$	$[\![\phi_1^d \to \phi_2^d]\!]_{r\circ v}$
0	1	0	1	1	0	0	1
0	1	1	0	1	0	1	0
1	0	0	1	0	1	0	1
1	0	1	0	1	0	0	1

A tautology $\phi \in \mathcal{F}_E$ has the valuation $[\![\phi]\!]_v = 1$ for all primitive valuations v. Therefore $[\![\phi]\!]_{r\circ v} = 1$ also holds, since $r \circ v$ is also a primitive valuation. Thus, $[\![\phi]\!]_{r\circ v} = r([\![\phi^d]\!]_{r\circ r\circ v}) = r([\![\phi^d]\!]_v)$, and $[\![\phi^d]\!]_v = 0$. Therefore ϕ^d is a contradiction. Argument also follows vice versa. □

Natural Deduction System We define a dual natural deduction system by interpreting all premises, assumptions and conclusions of the rules of the standard natural deduction system $\mathbf{Nc}$ by the duality interpretation $\cdot^d$, below. For every deduction in $\mathbf{Nc}$, there exists a dual deduction in $\mathbf{Nc}^d$.

Definition 4.13. A proof tree of $\mathbf{Nc}^d$ with respect to $\mathcal{F}_E$ is constructed inductively, with the rules:

$$\frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi & \psi\end{matrix}}{\phi \vee \psi} \vee_i \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \vee \psi\end{matrix}}{\phi} \vee_{e_1} \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi \vee \psi\end{matrix}}{\psi} \vee_{e_2} \qquad \frac{\begin{matrix}\mathcal{D} \\ \phi\end{matrix}}{\phi \wedge \psi} \wedge_{i_1} \qquad \frac{\begin{matrix}\mathcal{D} \\ \psi\end{matrix}}{\phi \wedge \psi} \wedge_{i_2} \qquad \frac{\begin{matrix} & [\phi]^u & [\psi]^w \\ \mathcal{D} & \mathcal{D}' & \mathcal{D}'' \\ \phi \wedge \psi & \chi & \chi\end{matrix}}{\chi} \wedge_e, u, w$$

$$\frac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \psi\end{matrix}}{\phi \nleftarrow \psi} \nleftarrow_i, u \qquad \frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi \nleftarrow \psi & \phi\end{matrix}}{\psi} \nleftarrow_e \qquad \frac{\begin{matrix}\mathcal{D} \\ \top\end{matrix}}{\phi} \top_e \qquad \frac{\begin{matrix}\mathcal{D} & \mathcal{D}' \\ \phi & \neg\phi\end{matrix}}{\top} \neg_e \qquad \frac{\begin{matrix}[\phi]^u \\ \mathcal{D} \\ \top\end{matrix}}{\neg\phi} \neg_i, u \qquad \frac{}{\phi \wedge \neg\phi} \text{LNC} \qquad \frac{}{\bot} \bot_i$$

However, this natural deduction system $\mathbf{Nc}^d$ is not able to derive formulas constructed with $\to$, which is a serious drawback. This system is also not sound and complete with respect to semantic entailment, since we have already seen that $\cdot^d$ is not truth-preserving.

Proposition 4.14. *The derivation* $\vdash_{\text{Ncd}} \phi$ *holds if and only if* ϕ *is a contradiction.*

Proof. Consider towards absurdity, that $\vdash_{\text{Ncd}} \phi$ holds but ϕ is not a contradiction or ϕ is a contradiction but $\vdash_{\text{Ncd}} \phi$ does not hold. We know that if ϕ is a tautology then ϕ^d is a contradiction and that if ϕ is a contradiction then ϕ^d is a tautology by Proposition 4.12. Since we merely interpreted the rules of $\mathbf{Nc}$ into $\mathbf{Nc}^d$, this would imply that $\vdash_{\text{Nc}} \phi$ holds but ϕ is not a tautology or ϕ is a tautology but $\vdash_{\text{Nc}} \phi$ does not hold. That contradicts soundness or completeness of $\mathbf{Nc}$, respectively. □

The following result is not very surprising: the system $\mathbf{Nc}$ is similar to the system $\mathbf{Nc}^d$.

Proposition 4.15. $\mathbf{Nc} \simeq \mathbf{Nc}^d$

Proof. Follows from the construction of $\mathbf{Nc}^d$. □

Given a tautology ϕ, we obtain the contradiction ϕ^d by Proposition 4.12. Then $\neg\phi^d$ is a tautology again. A natural deduction system that is complete with respect to entailment in the classical sense must derive any such negated-dual tautology. We will use this observation in the next section, where we try to obtain a natural deduction system that is sound and complete with respect to entailment in this sense. In general we suspect the following:

Conjecture 4.16. *Given a natural deduction system* $\mathbf{D}$ *and a suitable duality interpretation* $\cdot^d$*,* $\mathbf{D}$ *is only complete with respect to classical semantics if it holds that* $\phi_1, \ldots, \phi_n \vdash \psi$ *if and only if* $\neg\phi_1^d, \ldots, \neg\phi_n^d \vdash \neg\psi^d$.

Even more general, we suspect the following:

Conjecture 4.17. *A given natural deduction system* $\mathbf{D}$ *is only sound and complete with respect to classical semantics if for all truth-preserving interpretations* t *it holds that* $\phi_1, \ldots, \phi_n \vdash \psi$ *if and only if* $t(\phi_1), \ldots, t(\phi_n) \vdash t(\psi)$.

TOP (α)	BOT (α)	IMP1	NIB1	IMP2	NIB2	NOR	NAND	NOT (β)	ID (β)
$\alpha \to \alpha$	$\alpha \nleftarrow \alpha$	$\alpha \to \beta$	$\alpha \nleftarrow \beta$	$\beta \to \alpha$	$\beta \nleftarrow \alpha$	$\alpha \nleftarrow (\beta \to \alpha)$	$\alpha \to (\beta \nleftarrow \alpha)$	$\alpha \nleftarrow (\alpha \to \beta)$	$(\alpha \to \beta) \to \alpha$
						$\beta \nleftarrow (\alpha \to \beta)$	$\beta \to (\alpha \nleftarrow \beta)$	$\alpha \to (\alpha \nleftarrow \beta)$	$(\alpha \nleftarrow \beta) \nleftarrow \alpha$
								$\alpha \nleftarrow \top$	$\top \to \alpha$
								$\alpha \to \bot$	$\bot \nleftarrow \alpha$
									α

OR	AND	XOR	EQUIV
$(\alpha \to \beta) \to \beta$	$(\alpha \nleftarrow \beta) \nleftarrow \beta$	$(\alpha \to \beta) \to (\alpha \nleftarrow \beta)$	$(\alpha \nleftarrow \beta) \nleftarrow (\alpha \to \beta)$
$(\beta \to \alpha) \to \alpha$	$(\beta \nleftarrow \alpha) \nleftarrow \alpha$	$(\beta \to \alpha) \to (\beta \nleftarrow \alpha)$	$(\beta \nleftarrow \alpha) \nleftarrow (\beta \to \alpha)$

Figure 3: Equivalence classes, dummy variable in parenthesis, all 16 binary functions.

4.3 Arrows

We will investigate a system consisting only of implication and not-implied-by. Let the set of connectives be $A = \{\to, \nleftarrow\}$. The set of alternative formulas $\mathcal{F}_A$ is a subset of the extended formulas $\mathcal{F}_E$ of the previous section. The main idea presented in this section is that we combine the techniques of minimal formulas and duality to form a new natural deduction system. We show an *interpretation* of standard formulas as alternative formulas and a *natural deduction system*, called $\mathbf{Na}$, for alternative deduction (or arrow deduction). Finally, we prove the soundness and completeness with respect to classical semantic entailment of this alternative system by *similarity* with $\mathbf{Nm}^{(+\mathrm{DN})}$.

Intuition We first provide some intuition working only with $\to$ and $\nleftarrow$. Intuitionistic logic, and the Brouwer–Heyting–Kolmogorov interpretation to be more precise, interprets the implication connective as the existence of a function that transforms a proof of the left direct subformula into a proof of the right direct subformula, see [GLT89] and [Tro99]. In Definition 4.2, we translate $\phi \vee \psi$ into $(\phi \to \psi) \to \psi$, which bears the intuition that a disjunction is a nested transformation: we can transform a function of a proof of ϕ that results in a proof of ψ into a proof of ψ. If we already have a proof of ψ, then we may always transform any other proof into a proof of ψ. And in the other case, if we already have a proof of ϕ, then we may transform it into a proof of ψ by the assumed nested transformation. Thus, either ϕ is provable or ψ is provable (the intuitionistic interpretation of disjunction).

The dual of $\phi \vee \psi$ is $\phi \wedge \psi$, which is translated by Definition 4.2 into a formula that is not easily explained. However, in Definition 4.18, $\phi \wedge \psi$ is translated as $(\phi \nleftarrow \psi) \nleftarrow \psi$. Intuitively, we may understand $\phi \nleftarrow \psi$ as a proof of a counter-example of the implication $\psi \to \phi$, i.e. that no function exists that transforms a proof of ψ into a proof of ϕ. We need to provide a counter-example to the implication $\psi \to (\phi \nleftarrow \psi)$, thus we need a concrete example where we prove that ψ holds and that $\phi \nleftarrow \psi$ does not hold. To prove that $\phi \nleftarrow \psi$ does not hold, we need to provide a function of $\psi \to \phi$. Since we already have the proof that ψ holds, we still require a proof of ϕ. Therefore, to prove $(\phi \nleftarrow \psi) \nleftarrow \psi$ we need both a proof of ϕ and a proof of ψ (the intuitionistic interpretation of conjunction).

Interpretation Alternative formulas admit a classical interpretation given by Definition 3.4. Before we consider any other interpretations, we observe the equivalence classes in Figure 3. For all $\alpha, \beta \in \mathcal{F}_A$, the formulas shown in columns are equivalent. Note that dummy variables do not have any influence on a valuation, i.e. top and bottom always evaluate to a certain value regardless of α. The column of ID is remarkable since it contains both direct sub-formulas of Peirce Law we saw earlier, viz. $(\alpha \to \beta) \to \alpha$ is equivalent to α.

Since top and bottom are independent of its dummy variable, we will make use of an arbitrary $\chi \in \mathcal{F}_{A \setminus \{\top,\bot,\neg\}}$. It seems easier to define a countably infinite number of interpretations than to introduce a formal notion of some "do not care"-symbol. We define an interpretation of standard formulas as alternative formulas in Definition 4.18.

This interpretation is idempotent, i.e. $(\phi^a)^a = \phi^a$. Note that this interpretation is also truth-preserving, i.e. for all $\phi \in \mathcal{F}_S$ it holds that $\phi \equiv \phi^a$. For formulas in $\mathcal{F}_A$ we also use the symbols: $\bot := \chi \nleftarrow \chi$, $\top := \chi \to \chi$ and for any $\phi \in \mathcal{F}_A$ we let $\neg\phi := \phi \to (\chi \nleftarrow \chi)$.

Definition 4.18. Given an arbitrary $\chi \in \mathcal{F}_A$, called the dummy formula. For any formula ϕ we inductively construct an alternative formula ϕ^a by $\cdot^a : \mathcal{F}_C \to \mathcal{F}_D$, with respect to some $C \subseteq \{\top, \bot, \neg, \wedge, \vee, \to, \nleftarrow\}$:

$$\begin{aligned}
\phi^a &:= \phi && \text{for all } \phi \in \mathcal{A},\\
\top^a &:= \chi \to \chi, &&\\
\bot^a &:= \chi \nleftarrow \chi, &&\\
(\neg\phi)^a &:= \phi^a \to (\chi \nleftarrow \chi) && \text{for all } \phi \in \mathcal{F}_C,\\
(\phi \vee \psi)^a &:= (\phi^a \to \psi^a) \to \psi^a && \text{for all } \phi, \psi \in \mathcal{F}_C,\\
(\phi \wedge \psi)^a &:= (\phi^a \nleftarrow \psi^a) \nleftarrow \psi^a && \text{for all } \phi, \psi \in \mathcal{F}_C,\\
(\phi \to \psi)^a &:= \phi^a \to \psi^a && \text{for all } \phi, \psi \in \mathcal{F}_C,\\
(\phi \nleftarrow \psi)^a &:= \phi^a \nleftarrow \psi^a && \text{for all } \phi, \psi \in \mathcal{F}_C.
\end{aligned}$$

Natural Deduction System We define a natural deduction system with rules for the introduction and elimination for implication and not-implied-by below.

Definition 4.19. A proof tree of $\mathbf{Na}$ with respect to $\mathcal{F}_A$ is constructed inductively with the rules:

$$\dfrac{\begin{matrix}[\phi]^u\\ \mathcal{D}\\ \psi\end{matrix}}{\phi \to \psi}\to_i, u \qquad \dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}'\\ \phi \to \psi & \phi\end{matrix}}{\psi}\to_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg(\phi \to \psi)\end{matrix}}{\phi}\to_{e_2}$$

$$\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}'\\ \neg\phi & \psi\end{matrix}}{\phi \nleftarrow \psi}\nleftarrow_i \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \phi \nleftarrow \psi\end{matrix}}{\neg\phi}\nleftarrow_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \phi \nleftarrow \psi\end{matrix}}{\psi}\nleftarrow_{e_2}$$

Before we continue, we explain the significance of the rule $\to_{e_2}$. It is a generalization of double negation elimination, the axiom in $\mathbf{Nm}^{(+\mathrm{DN})}$ of Definitions 4.4 and 4.7, since if $\psi = \bot$, it allows us to deduce ϕ from $\neg\neg\phi$. We have obtained this rule by the suggestion of Conjecture 4.16. We verify that the following rules are derivable in $\mathbf{Na}$:

$$\dfrac{\begin{matrix}[\neg\phi^d]^u\\ \mathcal{D}\\ \neg\psi^d\end{matrix}}{\neg(\phi^d \nleftarrow \psi^d)}\to_i^d, u \qquad \dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}'\\ \neg(\phi^d \nleftarrow \psi^d) & \neg\phi^d\end{matrix}}{\neg\psi^d}\to_{e_1}^d \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg\neg(\phi^d \nleftarrow \psi^d)\end{matrix}}{\neg\phi^d}\to_{e_2}^d$$

$$\dfrac{\begin{matrix}\mathcal{D} & \mathcal{D}'\\ \neg\neg\phi^d & \neg\psi^d\end{matrix}}{\neg(\phi^d \to \psi^d)}\nleftarrow_i^d \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg(\phi^d \to \psi^d)\end{matrix}}{\neg\neg\phi^d}\nleftarrow_{e_1}^d \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg(\phi^d \to \psi^d)\end{matrix}}{\neg\psi^d}\nleftarrow_{e_2}^d$$

First, we can get rid of all $\cdot^d$ markers, since a rule applies to all formulas, including dual formulas. Additionally, let $n : \mathcal{F}_A \rightarrow \mathcal{F}_A$ be a truth-preserving, identity interpretation except for $n(\neg\neg\phi) := \phi$. It eliminates double negations, which according to Conjecture 4.17 is a necessary condition for soundness and completeness. We now verify that in $\mathbf{Na}$ the following rules are derivable:

$$\dfrac{\begin{matrix}[\neg\phi]^u\\ \mathcal{D}\\ \neg\psi\end{matrix}}{\neg(\phi \not\leftarrow \psi)}\rightarrow_i^d, u \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg(\phi\not\leftarrow\psi)\end{matrix} \quad \begin{matrix}\mathcal{D}'\\ \neg\phi\end{matrix}}{\neg\psi}\rightarrow^d_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \phi\not\leftarrow\psi\end{matrix}}{\neg\phi}\rightarrow^d_{e_2}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \phi\end{matrix} \quad \begin{matrix}\mathcal{D}'\\ \neg\psi\end{matrix}}{\neg(\phi\rightarrow\psi)}\not\leftarrow^d_i \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg(\phi\rightarrow\psi)\end{matrix}}{\phi}\not\leftarrow^d_{e_1} \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \neg(\phi\rightarrow\psi)\end{matrix}}{\neg\psi}\not\leftarrow^d_{e_2}$$

Again we assume in all following proofs that there is an unlimited supply of unused distinct markers $u_1, u_2, \ldots$. The rule $\rightarrow^d_i$ is derivable in $\mathbf{Na}$:

$$\dfrac{\dfrac{\dfrac{\phi\not\leftarrow\psi^{u_1}}{\psi}\not\leftarrow_{e_2} \quad \dfrac{\dfrac{\begin{matrix}[\neg\phi]^u\\ \mathcal{D}\\ \neg\psi\end{matrix}}{\neg\phi\rightarrow\neg\psi}\rightarrow_i, u \quad \dfrac{\phi\not\leftarrow\psi^{u_1}}{\neg\phi}\not\leftarrow_{e_1}}{\neg\psi}\rightarrow_e}{\bot}\rightarrow_e}{\neg(\phi\not\leftarrow\psi)}\rightarrow_i, u_1$$

The rule $\rightarrow^d_{e_1}$ is derivable in $\mathbf{Na}$:

$$\dfrac{\dfrac{\neg(\phi\not\leftarrow\psi)^{u_2} \quad \dfrac{\neg\phi^{u_3} \quad \psi^{u_1}}{\phi\not\leftarrow\psi}\not\leftarrow_i}{\bot}\rightarrow_e}{\neg\psi}\rightarrow_i, u_1$$

The rule $\rightarrow^d_{e_2}$ is trivially derivable in $\mathbf{Na}$. The rule $\not\leftarrow^d_i$ is derivable in $\mathbf{Na}$:

$$\dfrac{\dfrac{\neg\psi^{u_2} \quad \dfrac{\phi\rightarrow\psi^{u_1} \quad \phi^{u_3}}{\psi}\rightarrow_e}{\bot}\rightarrow_e}{\neg(\phi\rightarrow\psi)}\rightarrow_i, u_1$$

The rule $\not\leftarrow^d_{e_1}$ is trivially derivable in $\mathbf{Na}$. Indeed, we have introduced this rule precisely because we failed to derive it. The rule $\not\leftarrow^d_{e_2}$ is derivable in $\mathbf{Na}$:

$$\dfrac{\dfrac{\neg(\phi\rightarrow\psi)^{u_2} \quad \dfrac{\psi^{u_1}}{\phi\rightarrow\psi}\rightarrow_i}{\bot}\rightarrow_e}{\neg\psi}\rightarrow_i, u_1$$

Similarity To show the soundness and completeness of $\mathbf{Na}$, we show that the interpretation of Definition 4.2, partially reproduced below, is provability-preserving and thus suitable for proving the similarity $\mathbf{Na} \simeq \mathbf{Nm}^{(+\mathrm{DN})}$. We also partially reproduce the interpretation $a : \mathcal{F}_C \to \mathcal{F}_A$ with respect to some χ below. Again it seems easier to define a countably infinite number of minimal formula interpretations.

Definition. For any formula ϕ we inductively construct a formula $\phi^\dagger$ by $\cdot^\dagger : \mathcal{F}_A \to \mathcal{F}_M$:

$$
\begin{aligned}
(\phi \to \psi)^\dagger &:= \phi^\dagger \to \psi^\dagger && \text{for all } \phi, \psi \in \mathcal{F}_C, \\
(\phi \nleftarrow \psi)^\dagger &:= (\psi^\dagger \to \phi^\dagger) \to \bot && \text{for all } \phi, \psi \in \mathcal{F}_C.
\end{aligned}
$$

Definition. For any formula ϕ we inductively construct a formula ϕ^a by $\cdot^a : \mathcal{F}_M \to \mathcal{F}_A$ with respect to the dummy variable $\chi \in \mathcal{F}_A$:

$$
\begin{aligned}
\bot^a &:= \ \chi \nleftarrow \chi, \\
(\phi \to \psi)^a &:= \phi^a \to \psi^a && \text{for all } \phi, \psi \in \mathcal{F}_M.
\end{aligned}
$$

Both interpretations are truth-preserving.

Theorem 4.20. $\mathbf{Na} \simeq \mathbf{Nm}^{(+\mathrm{DN})}$

Proof. ($\Rightarrow$) We show that $\cdot^\dagger$ is a provability-preserving interpretation from $\mathbf{Na}$ with respect to χ to $\mathbf{Nm}^{(+\mathrm{DN})}$. Let $\chi^\dagger$ be the interpretation of χ. Given a derivation $\Gamma \vdash \psi$ of $\mathbf{Na}$, we show a derivation $\Gamma^\dagger \vdash \psi^\dagger$ of $\mathbf{Nm}^{(+\mathrm{DN})}$, by showing that all rules of the former are derivable in the latter, i.e. we show only one case of the similarity $\mathbf{Na}^\dagger \sim \mathbf{Nm}^{(+\mathrm{DN})}$, where, given a derivation $\Gamma \vdash \psi$ of $\mathbf{Na}^\dagger$, we show a derivation $\Gamma \vdash \psi$ of $\mathbf{Nm}^{(+\mathrm{DN})}$.

Rules $\to_i$, $\to_{e_1}$ are trivial. For the rule $\to_{e_2}$ of $\mathbf{Na}$, the interpretation $\to_{e_2}^\dagger$ of $\mathbf{Na}^\dagger$ with conclusion $\phi^\dagger$ and assumption $(\phi^\dagger \to \psi^\dagger) \to (\chi^\dagger \to \chi^\dagger) \to \bot$ is derivable in $\mathbf{Nm}^{(+\mathrm{DN})}$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{DN}
\UnaryInfC{$((\phi^\dagger \to \bot) \to \bot) \to \phi^\dagger$}
\AxiomC{$((\phi \nleftarrow \psi)^\dagger)^{u_5}$}
\AxiomC{$(\phi^\dagger \to \psi^\dagger)^{u_2}$}
\RightLabel{$\to_e$}
\BinaryInfC{$(\chi^\dagger \to \chi^\dagger) \to \bot$}
\AxiomC{$(\chi^\dagger)^{u_3}$}
\RightLabel{$\to_i, u_3$}
\UnaryInfC{$\chi^\dagger \to \chi^\dagger$}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i, u_2$}
\UnaryInfC{$(\phi^\dagger \to \psi^\dagger) \to \bot$}
\AxiomC{}
\RightLabel{DN}
\UnaryInfC{$((\psi^\dagger \to \bot) \to \bot) \to \psi^\dagger$}
\AxiomC{$\phi^\dagger \to \bot^{u_1}$}
\AxiomC{$(\phi^\dagger)^{u_4}$}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i$}
\UnaryInfC{$(\psi^\dagger \to \bot) \to \bot$}
\RightLabel{$\to_e$}
\BinaryInfC{$\psi^\dagger$}
\RightLabel{$\to_i, u_4$}
\UnaryInfC{$\phi^\dagger \to \psi^\dagger$}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i, u_1$}
\UnaryInfC{$(\phi^\dagger \to \bot) \to \bot$}
\RightLabel{$\to_e$}
\BinaryInfC{$\phi^\dagger$}
\end{prooftree}
$$

For the rule $\nleftarrow_i$ of $\mathbf{Na}$, the interpretation $\nleftarrow_i^\dagger$ of $\mathbf{Na}^\dagger$ with conclusion $(\psi^\dagger \to \phi^\dagger) \to \bot$ and assumptions $\phi^\dagger \to (\chi^\dagger \to \chi^\dagger) \to \bot$ and $\psi^\dagger$ is derivable in $\mathbf{Nm}^{(+\mathrm{DN})}$:

$$
\begin{prooftree}
\AxiomC{$((\neg\phi)^\dagger)^{u_4}$}
\AxiomC{$(\phi^\dagger)^{u_2}$}
\RightLabel{$\to_e$}
\BinaryInfC{$(\chi^\dagger \to \chi^\dagger) \to \bot$}
\AxiomC{$(\chi^\dagger)^{u_3}$}
\RightLabel{$\to_i, u_3$}
\UnaryInfC{$\chi^\dagger \to \chi^\dagger$}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i, u_2$}
\UnaryInfC{$\phi^\dagger \to \bot$}
\AxiomC{$(\psi^\dagger \to \phi^\dagger)^{u_1}$}
\AxiomC{$(\psi^\dagger)^{u_5}$}
\RightLabel{$\to_e$}
\BinaryInfC{$\phi^\dagger$}
\RightLabel{$\to_e$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_i, u_1$}
\UnaryInfC{$(\psi^\dagger \to \phi^\dagger) \to \bot$}
\end{prooftree}
$$

For the rule $\nleftarrow_{e_1}$ of $\mathbf{Na}$, the interpretation $\nleftarrow_{e_1}^{\dagger}$ of $\mathbf{Na}^{\dagger}$ with conclusion $\phi^{\dagger} \to (\chi^{\dagger} \to \chi^{\dagger}) \to \bot$ and assumption $(\psi^{\dagger} \to \phi^{\dagger}) \to \bot$ is derivable in $\mathbf{Nm}^{(+\mathrm{DN})}$:

$$\dfrac{\dfrac{\dfrac{((\psi^{\dagger} \to \phi^{\dagger}) \to \bot)^{u_2} \qquad \dfrac{\phi^{u_1}}{\psi^{\dagger} \to \phi^{\dagger}}\,\to_i}{\bot}\,\to_e}{(\chi^{\dagger} \to \chi^{\dagger}) \to \bot}\,\to_i}{\phi^{\dagger} \to (\chi^{\dagger} \to \chi^{\dagger}) \to \bot}\,\to_i, u_1$$

For the rule $\nleftarrow_{e_2}$ of $\mathbf{Na}$, the interpretation $\nleftarrow_{e_2}^{\dagger}$ of $\mathbf{Na}^{\dagger}$ with conclusion $\psi^{\dagger}$ and assumption $(\psi^{\dagger} \to \phi^{\dagger}) \to \bot$ is derivable in $\mathbf{Nm}^{(+\mathrm{DN})}$:

$$\dfrac{\dfrac{}{((\psi^{\dagger} \to \bot) \to \bot) \to \psi^{\dagger}}\,\mathrm{DN} \qquad \dfrac{\dfrac{(\psi^{\dagger} \to \phi^{\dagger}) \to \bot \qquad \dfrac{\dfrac{\dfrac{}{((\phi^{\dagger} \to \bot) \to \bot) \to \phi^{\dagger}}\,\mathrm{DN} \qquad \dfrac{\dfrac{\psi^{\dagger} \to \bot^{u_1} \quad (\psi^{\dagger})^{u_2}}{\bot}\,\to_e}{(\phi^{\dagger} \to \bot) \to \bot}\,\to_i}{\phi^{\dagger}}\,\to_e}{\psi^{\dagger} \to \phi^{\dagger}}\,\to_i, u_2}{\bot}\,\to_e}{(\psi^{\dagger} \to \bot) \to \bot}\,\to_i, u_1}{\psi^{\dagger}}\,\to_e$$

($\Leftarrow$) We show that $\cdot^{a}$ with respect to χ is a provability-preserving interpretation from $\mathbf{Nm}^{(+\mathrm{DN})}$ to $\mathbf{Na}$. Given a derivation $\Gamma \vdash \psi$ of $\mathbf{Nm}^{(+\mathrm{DN})}$, we show a derivation $\Gamma^{a} \vdash \psi^{a}$ of $\mathbf{Na}$, by showing that all rules of the former are derivable in the latter, i.e. we show only one case of the similarity $(\mathbf{Nm}^{(+\mathrm{DN})})^{a} \sim \mathbf{Na}$, where, given a derivation $\Gamma \vdash \psi$ of $(\mathbf{Nm}^{(+\mathrm{DN})})^{a}$, we show a derivation $\Gamma \vdash \psi$ of $\mathbf{Na}$.

Rules $\to_i, \to_{e_1}$ are trivial. For axiom DN of $\mathbf{Nm}^{(+\mathrm{DN})}$, the interpretation DN^{a} of $(\mathbf{Nm}^{(+\mathrm{DN})})^{a}$ with conclusion $((\phi^{a} \to \chi \nleftarrow \chi) \to \chi \nleftarrow \chi) \to \phi^{a}$ is derivable in $\mathbf{Na}$:

$$\dfrac{\dfrac{)\neg\neg\phi^{a})^{u_1}}{\phi^{a}}\,\to_{e_2}}{\neg\neg\phi^{a} \to \phi^{a}}\,\to_i, u_1$$

□

The following theorem follows from similarity, and is not surprising now we have found out that $\nleftarrow_i$, $\nleftarrow_{e_1}$ and $\nleftarrow_{e_2}$ of $\mathbf{Na}$ are derivable (and thus admissible) rules of $\mathbf{Nm}^{(+\mathrm{DN})}$ under a certain truth-preserving interpretation.

Theorem 4.21. *The natural deduction system* $\mathbf{Na}$ *is sound and complete with respect to classical semantic entailment.*

5 Conclusion

We have given an alternative, constructive proof of the known result that in minimal and intuitionistic propositional logic one can derive classical tautologies under the admission of certain axioms: double negation elimination and Peirce's Law respectively, in Section 4.1. We have shown that there exists a dual natural deduction system that derives only all classical contradictions, and that a classical deduction system and its dual are closely related with respect to classical semantics, in Section 4.2.

We have introduced an alternative natural deduction system for an alternative set of connectives consisting of implication ($\rightarrow$) and not-implied-by ($\not\leftarrow$), we have investigated the dual of the alternative system, and we have proved the soundness and completeness of the alternative system by showing truth-preserving and provability-preserving interpretations to minimal logic with the axiom of double negation elimination and applying the previously known results of soundness and completeness of that logic with respect to classical semantics, in Section 4.3.

Future Work   Are Conjectures 4.16 and 4.17 true? Is the condition of Conjecture 4.17 not only necessary but also sufficient?

Bibliography

[AH03] Zena M. Ariola and Hugo Herbelin. Minimal classical logic and control operators. In *Automata, Languages and Programming*, pages 871–885. Springer, 2003.

[D'A99] Marcello D'Agostino. Tableau methods for classical propositional logic. In *Handbook of tableau methods*, pages 45–123. Springer, 1999.

[Gen33] Gerhard Gentzen. Über die existenz unabhängiger axiomensysteme zu unendlichen satzsystemen. *Mathematische Annalen*, 107(1):329–350, 1933.

[GLT89] Jean-Yves Girard, Yves Lafont, and Paul Taylor. Proofs and types. In *Cambridge tracts in theoretical computer science*, volume 7. Cambridge university press, 1989.

[Gra05] Clemens Grabmayer. *Relating Proof Systems for Recursive Types*. PhD thesis, Vrije Universiteit, 2005.

[HR04] Michael Huth and Mark Ryan. *Logic in Computer Science: Modelling and reasoning about systems*. Cambridge University Press, second edition edition, 2004.

[Hun33] Edward V. Huntington. Boolean algebra: a correction to "new sets of independent postulates for the algebra of logic...". *Transactions of the American Mathematical Society*, 35(2):557–558, 1933.

[ML78] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 1978.

[Pel99] Francis Jeffry Pelletier. A brief history of natural deduction. *History and Philosophy of Logic*, 20(1):1–31, 1999.

[Pos21] Emil L. Post. Introduction to a general theory of elementary propositions. *American Journal of Mathematics*, 43(3):163–185, 1921.

[RB12] Marc D. Riedel and Jehoshua Bruck. Cyclic boolean circuits. *Discrete Applied Mathematics*, 160(13):1877–1900, 2012.

[Smu95] Raymond M. Smullyan. *First-Order Logic*. Courier Corporation, 1995.

[Tro99] Anne Sjerp Troelstra. From constructivism to computer science. *Theoretical Computer Science*, 211(1):233–252, 1999.

[TS00] Anne Sjerp Troelstra and Helmut Schwichtenberg. *Basic Proof Theory*. Number 43. Cambridge University Press, 2000.

[VD83] Dirk Van Dalen. *Logic and structure*. Springer, 1983.

[WL11] Wei Wu and Zijian Li. A categorical approach to automated theorem proving (i). http://www.cs.zju.edu.cn/littlefugue/, 2011.

A Appendix

The following Java code is used to generate equivalence classes of Figure 3.

```
import java.io.File;
import java.io.PrintStream;
import java.util.HashMap;
import java.util.HashSet;
import java.util.Iterator;
import java.util.Map;
import java.util.Set;
public class NormalForms {
  static final class Valuation {
    private HashMap<Variable, Boolean> store = new HashMap<>();
    void set(Variable var, boolean val) {
      if (var == null) throw null;
      store.put(var, val);
    }
    boolean get(Variable var) { return store.get(var); }
  }
  static abstract class Formula {
    abstract boolean evaluate(Valuation primitive);
    public abstract String toString();
    abstract int size();
    abstract boolean equals(Formula other);
    public final boolean equals(Object obj) {
      if (obj == this) return true;
      if (obj instanceof Formula) return equals((Formula) obj);
      return false;
    }
    boolean isTop() { return false; }
    boolean isBottom() { return false; }
    boolean isSub(Formula sub) {
      return sub.equals(this);
    }
  }
  static class Variable extends Formula {
    String name;
    Variable(String name) {
      if (name == null) throw null;
      this.name = name;
    }
    boolean evaluate(Valuation primitive) {
      return primitive.get(this);
    }
    public String toString() { return name; }
    int size() { return 1; }
    boolean equals(Variable other) {
      return name.equals(other.name);
    }
    final boolean equals(Formula other) {
      if (other instanceof Variable)
        return equals((Variable) other);
      return false;
    }
    public int hashCode() { return name.hashCode(); }
  }
```

```
  static class Implication extends Formula {
    Formula left, right;
    Implication(Formula left, Formula right) {
      if (left == null || right == null) throw null;
      this.left = left;
      this.right = right;
    }
    boolean evaluate(Valuation primitive) {
      boolean l = left.evaluate(primitive), r = right.evaluate(primitive);
      return !l | r;
    }
    public String toString() {
      return "(" + left + "\\rightarrow␣" + right + ")";
    }
    int size() {
      return left.size() + 1 + right.size();
    }
    boolean equals(Implication other) {
      return left.equals(other.left) && right.equals(other.right);
    }
    final boolean equals(Formula other) {
      if (other instanceof Implication)
        return equals((Implication) other);
      return false;
    }
    public int hashCode() {
      int hash = 1;
      hash = hash * 31 + left.hashCode();
      hash = hash * 31 + right.hashCode();
      return hash;
    }
    boolean isTop() {
      return left.equals(right);
    }
    boolean isSub(Formula sub) {
      return super.isSub(sub) || left.isSub(sub) || right.isSub(sub);
    }
  }
  static class NonInitiation extends Formula {
    Formula left, right;
    NonInitiation(Formula left, Formula right) {
      if (left == null || right == null) throw null;
      this.left = left;
      this.right = right;
    }
    boolean evaluate(Valuation primitive) {
      boolean l = left.evaluate(primitive), r = right.evaluate(primitive);
      return !l & r;
    }
    public String toString() {
      return "(" + left + "\\not\\leftarrow␣" + right + ")";
    }
    int size() {
      return left.size() + 1 + right.size();
    }
    boolean equals(NonInitiation other) {
      return left.equals(other.left) && right.equals(other.right);
    }
```

```
      final boolean equals(Formula other) {
        if (other instanceof NonInitiation)
          return equals((NonInitiation) other);
        return false;
      }
      public int hashCode() {
        int hash = 1;
        hash = hash * 31 + left.hashCode();
        hash = hash * 31 + right.hashCode();
        return hash;
      }
      boolean isBottom() {
        return left.equals(right);
      }
      boolean isSub(Formula sub) {
        return super.isSub(sub) || left.isSub(sub) || right.isSub(sub);
      }
    }
    static abstract class Permutation implements
        Iterable<Formula>, Iterator<Formula> {
      public abstract void reset();
      public abstract boolean hasNext();
      public abstract Formula next();
      public final Iterator<Formula> iterator() { return this; }
    }
    static class VariablePermutation extends Permutation {
      int cur, max;
      VariablePermutation(int max) {
        if (max < 1) throw null;
        this.max = max;
      }
      public void reset() { cur = 0; }
      public boolean hasNext() { return cur < max; }
      public Variable next() {
        if (cur >= max) throw new IllegalStateException();
        cur++;
        return new Variable("a_" + cur);
      }
    }
    static class TreePermutation extends Permutation {
      Formula left, right;
      Permutation[] pers;
      int depth, cur;
      TreePermutation(int depth) {
        this(1, depth);
      }
      TreePermutation(int max, int depth) {
        if (depth < 1) throw null;
        this.depth = depth;
        pers = new Permutation[depth * 2];
        for (int i = 1; i <= depth; i++) {
          pers[i * 2 - 2] = (i == 1) ?
            new VariablePermutation(max) : new TreePermutation(max, i - 1);
          pers[i * 2 - 1] = (i == depth) ?
            new VariablePermutation(max) : new TreePermutation(max, depth-i);
        }
      }
```

```
    public void reset() {
      for (cur = depth; cur > 0; cur--) {
        pers[cur * 2 - 2].reset();
        pers[cur * 2 - 1].reset();
      }
    }
    public boolean hasNext() {
      if (right != null) return true;
      if (left != null) {
        if (pers[cur * 2 + 1].hasNext()) return true;
        left = null;
      }
      for (; cur < depth; cur++)
        if (pers[cur * 2].hasNext()) return true;
      return false;
    }
    public Formula next() {
      if (right != null) {
        Formula result = new NonInitiation(left, right);
        right = null;
        return result;
      }
      retry: while (true) {
        if (left != null) {
          if (pers[cur * 2 + 1].hasNext()) {
            right = pers[cur * 2 + 1].next();
            return new Implication(left, right);
          }
          left = null;
        }
        for (; cur < depth; cur++)
          if (pers[cur * 2].hasNext()) {
            left = pers[cur * 2].next();
            pers[cur * 2 + 1].reset();
            continue retry;
          }
        throw new IllegalStateException();
      }
    }
  }
  static class NormalPermutation extends Permutation {
    int max, depth;
    Permutation cur;
    NormalPermutation() { this(1); }
    NormalPermutation(int max) {
      if (max < 1) throw null;
      this.max = max;
      this.cur = new VariablePermutation(max);
    }
    public void reset() {
      cur = new VariablePermutation(max);
      depth = 0;
    }
    public boolean hasNext() { return true; }
```

```
    public Formula next() {
      while (true) {
        if (cur != null) {
          if (cur.hasNext())
            return cur.next();
          cur = null;
        }
        depth++;
        cur = new TreePermutation(max, depth);
      }
    }
  }
  static int count = 0;
  static Set<Formula> visited = new HashSet<>();
  static Map<String, Set<Formula>> groups = new HashMap<>();
  public static void main(String[] args) throws Exception {
    System.setOut(new PrintStream(new File("out.txt")));
    System.out.println("<table>");
    System.out.println("<thead>");
    System.out.println("<tr><th>Type</th><th>Formula</th><th></th><th>
        Minimal␣Formula</th><th>Truth-table</th></tr>");
    System.out.println("</thead>");
    System.out.println("<tbody>");
    printTableBody(3, 10);
    System.out.println("</tbody>");
    System.out.println("</table>");
  }
  static void printTableBody(int max, int depth) {
    String type = "\\alpha_{" + max + "," + depth + "}";
    perm: for (Formula f : new TreePermutation(max, depth)) {
      Set<Formula> form = getMinimalForms(f, max);
      if (form.isEmpty()) continue perm;
      form.add(f);
      for (Formula q : form) if (visited.contains(q)) continue perm;
      visited.addAll(form);
      String group = "\\Delta_{" + (++count) + "}";
      System.out.print("<tr>");
      System.out.print("<th>\\(");
      System.out.print(type);
      System.out.print("\\)</th><td>\\(");
      System.out.print(group);
      System.out.print("\\)</td><td>\\(=\\)</td>");
      System.out.print("<td>\\(\\{\\)");
      boolean comma = false;
      for (Formula a : form) {
        System.out.print("\\(");
        if (comma) System.out.print(',');
        System.out.print(a);
        System.out.print("\\)");
        comma = true;
      }
      System.out.print("\\(\\}\\)</td>");
      System.out.print("<td>");
      printTruth(f, max);
      System.out.print("<td>");
      System.out.println("</tr>");
    }
  }
```

```
  // Find some minimal form of provided formula; formula must only consist
  // of standard a1, ..., aN variables, provided some positive n <= 16.
  static Set<Formula> getMinimalForms(Formula form, int n) {
    if (n <= 0 || n > 16) throw new IllegalArgumentException();
    if (form == null) throw null;
    Set<Formula> result = new HashSet<Formula>();
    int size = form.size();
    for (Formula norm : new NormalPermutation(n)) {
      if (norm.size() > size)
        return result;
      if (isEquivalent(norm, form, n)) {
        if (norm.isBottom() || norm.isTop())
          return result;
        result.add(norm);
      }
    }
    throw new Error();
  }
  static boolean isEquivalent(Formula a, Formula b, int n) {
    Valuation v = new Valuation();
    Variable[] vars = new Variable[n];
    for (int j = 0; j < n; j++)
      vars[j] = new Variable("a_" + (j + 1));
    int ubound = 1 << n;
    for (int i = 0; i < ubound; i++) {
      int lbound = 1;
      for (int j = 0; j < n; j++) {
        v.set(vars[j], (i & lbound) == lbound);
        lbound <<= 1;
      }
      if (a.evaluate(v) != b.evaluate(v)) return false;
    }
    return true;
  }
  static void printTruth(Formula a, int n) {
    Valuation v = new Valuation();
    Variable[] vars = new Variable[n];
    for (int j = 0; j < n; j++) vars[j] = new Variable("a_" + (j + 1));
    int ubound = 1 << n;
    System.out.print("<table>");
    for (int i = 0; i < ubound; i++) {
      System.out.print("<tr>");
      int lbound = 1;
      for (int j = 0; j < n; j++) {
        System.out.print("<th>");
        boolean val = (i & lbound) == lbound;
        v.set(vars[j], val);
        System.out.print(val ? 't' : 'f');
        System.out.print("</th>");
        lbound <<= 1;
      }
      System.out.print("<td>");
      System.out.print(a.evaluate(v) ? 't' : 'f');
      System.out.print("</td></tr>");
    }
    System.out.print("</table>");
  }
}
```